

The main research topics we wish to pursue are problems in extremal graph theory. In particular, we are interested in problems from Ramsey Theory and Saturation Games.

**Ramsey Theory.** One of the classical areas of extremal combinatorics is Ramsey Theory, whose problems ask how large an object can be before it contain a certain structure. This theory was originally introduced by Ramsey to solve a problem in logic [5], and since then Ramsey theoretic results have appeared and been utilized in various areas of mathematics. Some notable Ramsey theoretic results includes Schur’s Theorem [6] in number theory (if  $\mathbb{N}$  is partitioned into finitely many classes, then some class contains  $x$ ,  $y$ , and  $z$  such that  $x + y = z$ ) and the Erdős-Szekeres theorem [1] in discrete geometry (if  $P$  is an infinite number of points in the plane in general position, then  $P$  contains infinitely many points in convex position). Ramsey Theory was a key tool in the proof of Gowers’ celebrated Dichtomy Theorem [3], a theorem in functional analysis.

Let  $K_n^{(k)}$  denote the complete  $k$ -uniform hypergraph on  $n$  vertices. Let  $r_k(n)$  denote the smallest  $N$  such that in any red/blue coloring of  $K_N^{(k)}$ , there exists either a red  $K_n^{(k)}$  or a blue  $K_n^{(k)}$ . A major problem in Ramsey Theory is to find bounds for  $r_k(n)$ . An extremal example of this is the case  $n = k + 1$ , for which the best known bounds are roughly  $2^k \leq r_k(k + 1) \leq 2^{2^k}$ .

In an attempt to improve the bounds on  $r_k(k + 1)$ , we consider a generalization of this problem. Let  $h_k(s)$  denote the smallest number  $N$  such that in any red/blue coloring of  $K_N^{(k)}$ , there either exists a red  $K_{k+1}^{(k)}$ , or there exists a set of  $k + 1$  vertices which contain at least  $s$  blue edges. For example,  $h_k(k + 1) = r_k(k + 1)$ . While it is not too difficult to prove that  $h_k(1)$  and  $h_k(2)$  are linear, the best known bounds for  $s = 3$  are roughly  $k + 2 \leq h_k(3) \leq 2^{2^k}$ . Our goal is to sharpen the bounds of  $h_k(s)$  for various  $s$ , with the hope that these methods could then be used to improve the bounds for  $r_k(k + 1)$ . As a first step, we would like to improve the bounds on  $h_k(3)$ , and we are currently exploring two different approaches to do this.

Our first approach comes from Design Theory. We will say that a  $k$ -uniform hypergraph on  $n$  vertices  $H$  is an  $(n, k, s)$ -Steiner system if every  $(k - 1)$ -set of  $H$  is contained in at least 1 edge but no more than  $s$  edges. Such objects are known as “almost Steiner systems,” with the case  $s = 1$  corresponding to the normal definition of a Steiner system. We were able to prove the following:  $h_k(s) > M$  if and only if there exists an  $(M, M - k, s - 1)$ -Steiner system. Using this and recent results of Keevash [4] from Design Theory, we were able to prove, for infinitely many  $k$ , that  $h_k(2) = k + \text{lpf}(k + 1)$ , where  $\text{lpf}(x)$  denotes the smallest prime factor of  $x$ . This result is somewhat surprising, as initially this problem does not seem to be related to the number theoretic properties of  $k$ . In a similar fashion, we suspect that the bounds of  $h_k(3)$  can be improved by looking at this problem from a Design Theory point of view.

Our second approach involves a relation between  $h_k(3)$  and the Turán number  $\text{ex}(n, K_{k+1}^{(k)})$ , which is the maximum number of edges a  $k$ -uniform hypergraph on  $n$  vertices can have without containing  $K_{k+1}^{(k)}$  as a subgraph. It can be shown that a significant improvement of the upper bound of  $\text{ex}(n, K_{k+1}^{(k)})$  would imply an improvement to the upper bound of  $h_k(3)$ . With Jacques Verstraete, we are exploring ideas using the probabilistic method to improve the upper bound on  $\text{ex}(n, K_{k+1}^{(k)})$ . Namely, we are working on a method to probabilistically find a large clique in a hypergraph by looking at the link hypergraph for each vertex. Improving the bounds on  $\text{ex}(n, K_{k+1}^{(k)})$ , in addition to being of independent interest, could improve the bounds for  $h_k(3)$ .

**Saturation Games.** Let  $\mathcal{F}$  be a set of (forbidden) graphs. Consider the following game played by two players, Min and Max, which we call the  $(\mathcal{F}, n)$ -game. The game starts with an empty graph  $G$  on  $n$  vertices. The players alternate turns adding edges to  $G$ , with the only restriction being that neither play can add an edge that would create some  $F \in \mathcal{F}$  as a subgraph in  $G$ , and

the game ends when no more edges can be added to  $G$ . If there are  $e$  edges in  $G$  at the end of the game, then Max's score for the game is  $e$  and Min's score is  $-e$ . Thus Max wants the game to last as long as possible and Min wants the game to end as quickly as possible. Assuming both players play optimally, we wish to bound the number of edges in  $G$  at the end of the game.

Let  $C_k$  denote the cycle graph on  $k$  vertices. Saturation Games were first introduced by Füredi et al. [2] in 1992, where they showed that the  $(\{C_3\}, n)$ -game ends with at least  $\frac{1}{2}n \log n$  edges. Using results from extremal graph theory, one can show that if  $\mathcal{F}$  contains any odd cycles, then the  $(\mathcal{F}, n)$ -game ends with at most  $\frac{1}{4}n^2$  edges. In terms of order of magnitude, these are the only known bounds for the number of edges at the end of the  $(\{C_3\}, n)$ -game. Despite this, we were able to show the following.

**Theorem 1.** [7] *For  $k \geq 2$ , the  $(\{C_3, C_5, \dots, C_{2k+1}\}, n)$ -game ends with quadratically many edges. Moreover, if  $k \geq 4$  the game ends with at most  $c_k n^2$  edges for some  $c_k < \frac{1}{4}$ .*

Saturation Games are still a relatively new subfield of extremal graph theory, and hence it contains many open problems that are likely to be solvable. In particular, we are interested in proving bounds for variants of the  $(\{C_3\}, n)$ -game. For example, Theorem 1 shows that the  $(\{C_3, C_5\}, n)$ -game ends with quadratically many edges, but it is not known whether the same is true for the  $(\{C_3, C_7\}, n)$ -game, even though these problems appear to be quite similar. Bounding these variant games should give insight as to what the correct order of magnitude is for the number of edges at the end of the  $(\{C_3\}, n)$ -game. Moreover, the techniques used to solve these variant games could potentially be used to improve the bounds of the  $(\{C_3\}, n)$ -game as well.

In addition to this, we would like to try to establish the foundations of a general theory for Saturation Games, which the field currently lacks. In particular, none of the currently known bounds for Saturation Game use the probabilistic method. Given how prominent the probabilistic method has been in solving various problems in extremal combinatorics, we believe that there exists Saturation Games whose bounds can be greatly improved by utilizing a probabilistic argument. Our plan is to develop elements of a probabilistic strategy that could be used by either Max or Min, and then to work out which  $(\mathcal{F}, n)$ -games these probabilistic strategies could be used for. Once probabilistic methods have been shown to work for some specific set of problems, we suspect that a more general theory for Saturation Games using the probabilistic method could be developed.

**Broader Impact.** We are excited to talk about our results at seminars and conferences, especially at seminars and conferences that we organize, such as the UCSD Graduate Student Combinatorics Seminar and the Graduate Student Combinatorics Conference 2020. In addition to this, we are scheduled to speak at the UCSD Combinatorics Seminar about our work in Saturation Games this winter.

#### REFERENCES

- [1] Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio mathematica*, 2:463–470, 1935.
- [2] Zoltán Füredi, Dave Reimer, and Ákos Seress. Hajnal's triangle-free game and extremal graph problems. *Congressus Numerantium*, pages 123–123, 1991.
- [3] W Timothy Gowers. A new dichotomy for banach spaces. *Geometric & Functional Analysis GAFA*, 6(6):1083–1093, 1996.
- [4] Peter Keevash. The existence of designs. *arXiv preprint arXiv:1401.3665*, 2014.
- [5] Frank P Ramsey. On a problem of formal logic. In *Classic Papers in Combinatorics*, pages 1–24. Springer, 2009.
- [6] Issai Schur. *Über die Existenz unendlich vieler Primzahlen in einigen speziellen arithmetischen Progressionen*. 1912.
- [7] Sam Spiro. Saturation games for odd cycles. *arXiv preprint arXiv:1808.03696*, 2018.