The Count of Monte Carlo

Sam Spiro, UC San Diego.

What is Zoom for Thought?

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The talks will be every week of (at least) Fall and Winter quarter, and for this quarter we will meet on Tuesdays at 2.

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- Talks need not include memes.

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Just let Vaki or me know if you'd like to give a talk on some specific day, or if you'd just like to be on the "reserve list."

Other Important things

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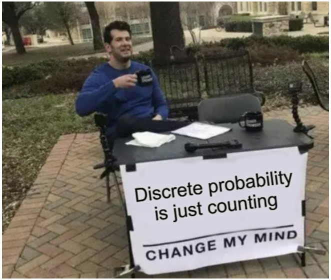
If you're a US citizen and a $\leq$ 2nd year, you should consider applying to the NSF GRFP (a fancy fellowship which gets you out of TAing for 3 years). Note that the deadline is October 22nd, and you can only apply once as a grad student. I have copies of my essays on my website (see point above), and there are several other recipients at UCSD that you can talk to about this.

Probability from Counting

Probability from Counting



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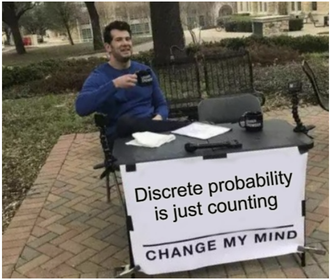

E.g. if I want to compute the probability of getting a given hand in poker, I can just reduce this to the enumeration problem of counting all the ways to get that given hand.

Probability from Counting

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As an aside, this talk is only about using probability to obtain (exact) enumerative combinatorics results, much more about using probability to get (approximate) extremal combinatorics results can be found in 261A.

Padlock Solitare

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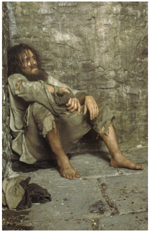
Situation: Edmond Dantés has been falsely accused of treason and imprisoned in Chateau D'If.



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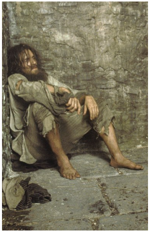

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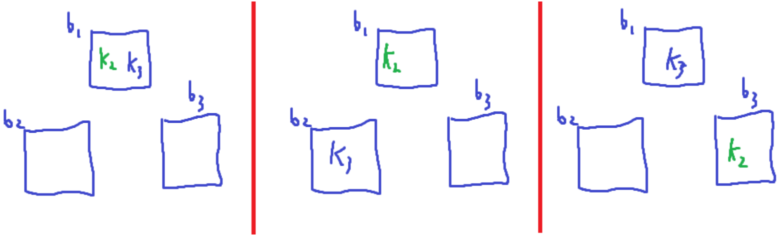
For $n = 3$, Edmond escapes if either (1) $k_2, k_3 \in b_1$, (2) $k_2 \in b_1$ and $k_3 \in b_2$, or (3) $k_3 \in b_1$ and $k_2 \in b_3$.



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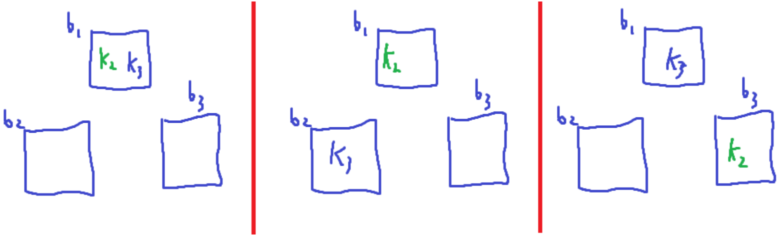

Since each of these three events are equally likely, we see that the probability of escape is $3/9 = 1/3$.

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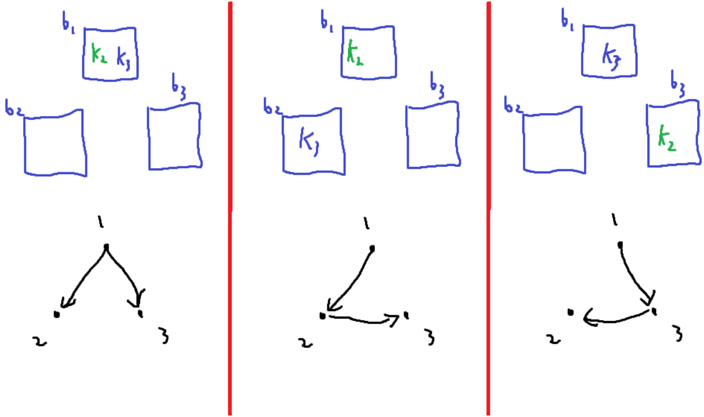

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$$\frac{t_n}{n^{n-1}} = \Pr[\text{Esacpe}] = \frac{1}{n} \implies t_n = n^{n-2}.$$

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Theorem (Cayley's Formula)

The number of labeled trees on n vertices is n^{n-2}.

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Using similar ideas one can count a lot of other things:

- Trees with a given degree sequence $d_1, \ldots, d_n$ (distribute keys uniformly conditional on each b_i having $d_i - 1$ keys).

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Many other variants can be found in the lovely paper by Wästlund (who also has a lot of other very nice papers).

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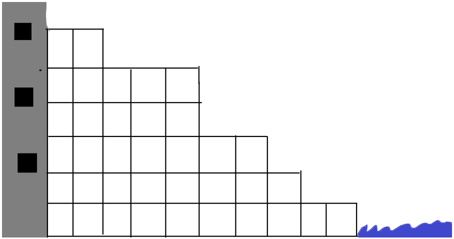
Unfortunately for Edmond, $n = 34$ in Chateau D'If, so it's pretty unlikely he'll escape with padlock solitaire. Fortunately for the Young man, his cell comes with a standard table, so he decides to carve a leg to use as a hook to dig himself out.

Hook Length Formula

Situation: Edmond is trapped in a dungeon whose cells are layed out in the picture below (or more generally some arrangement where the length of the rows decrease as you go down).



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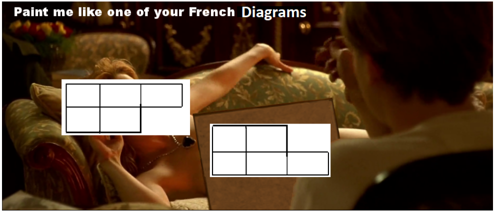

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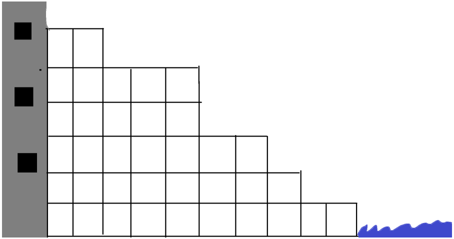
Edmond will break out of the dungeon using the following (random) procedure:



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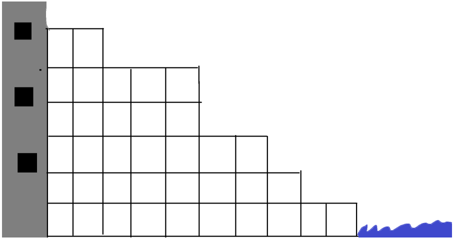

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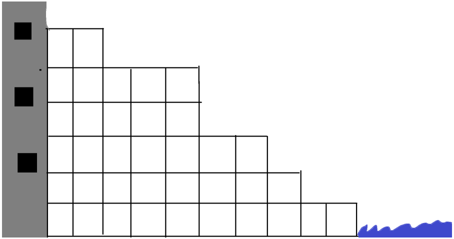

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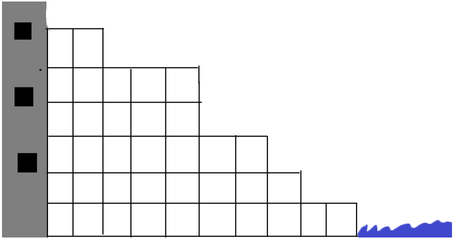

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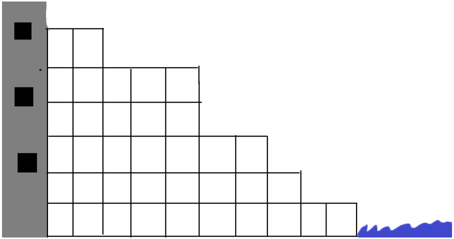

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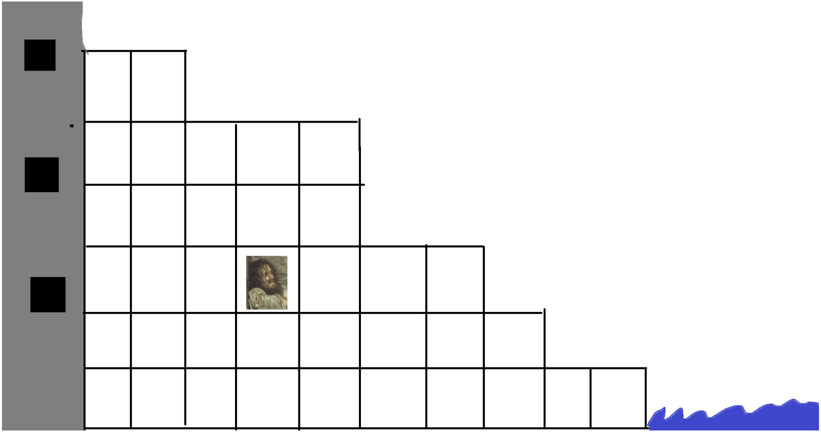

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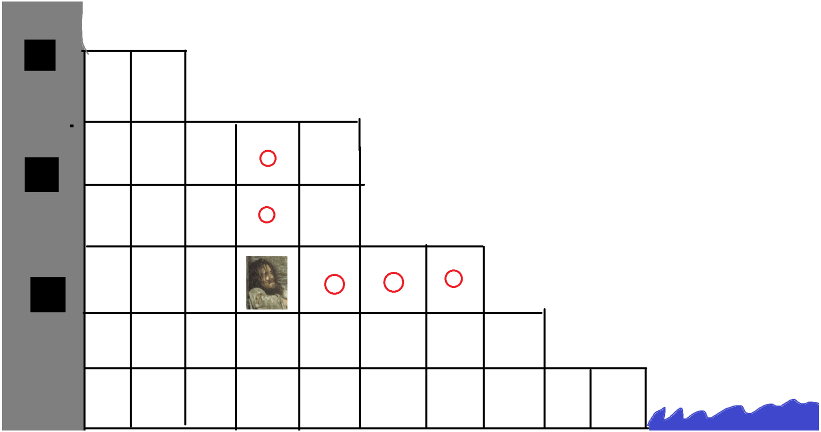

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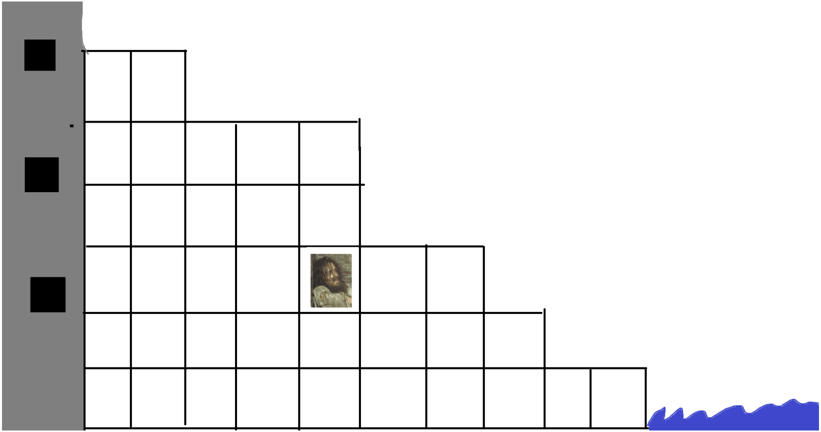

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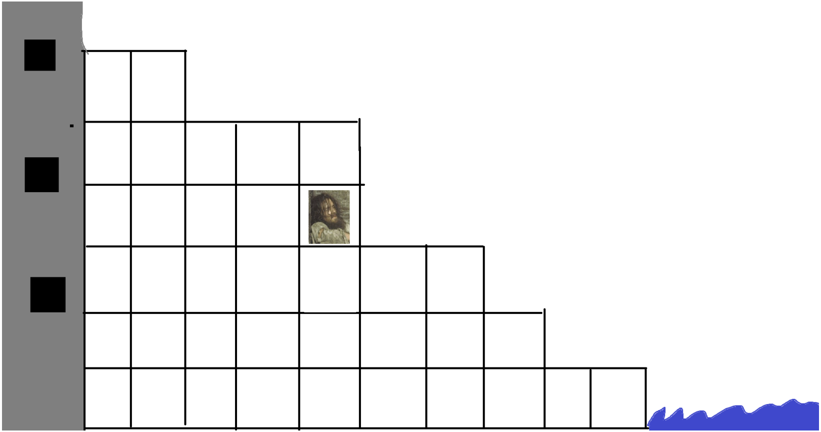

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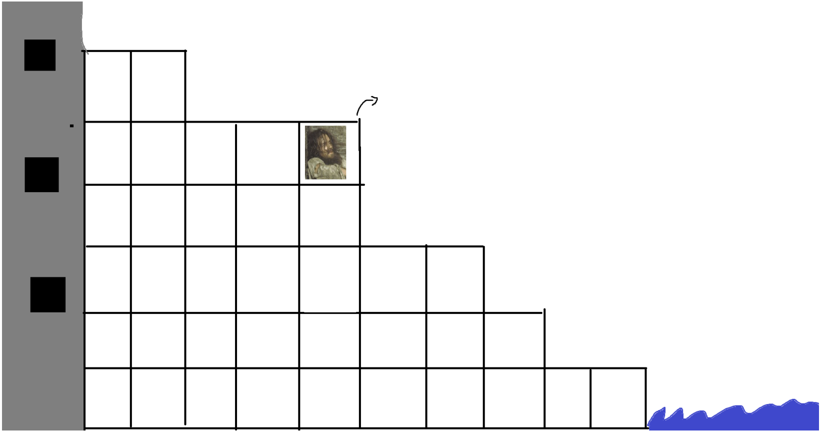

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Given λ, we define the hook length $h_{i,j}$ of a cell (i, j) to be the number of cells directly to the right or directly above (i, j) (with this including the cell (i, j) itself).

Hook Length Formula

Given a path $(a_1, b_1) \to (a_2, b_2) \to \cdots \to (a_m, b_m)$, we let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ denote the projections of this path.

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Lemma

Given cell (a, b), corner (α, β), and sets A, B, the probability that Edmond travels from (a, b) to (α, β) using a path with projections A, B is

$$\frac{1}{n} \prod_{i \in A \setminus \alpha} \frac{1}{h_{i,\beta} - 1} \prod_{j \in B \setminus \beta} \frac{1}{h_{\alpha,j} - 1}.$$

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Thus the probability of ending at a cell (α, β) is equal to the sum of these probabilities over all paths, and one can verify that this is equal to

$$\frac{1}{n} \prod_{1 \leq i < \alpha} \left(1 + \frac{1}{h_{i,\beta} - 1}\right) \prod_{1 \leq j < \beta} \left(1 + \frac{1}{h_{\alpha,j} - 1}\right).$$

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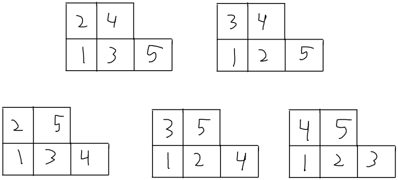

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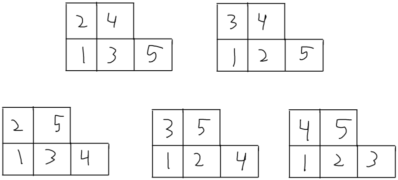

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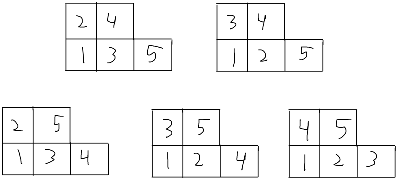


How many SYT of a given shape are there? This turns out to be an important question in representation theory and algebraic combinatorics, since this is the dimension of the irreducible representation of the symmetric group S_n indexed by λ.

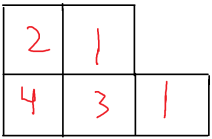
Hook Length Formula

Theorem (Hook length formula: Frame-Robinson-Thrall, Greene-Niejenhuis-Wilf)

If λ is a partition of n, then the number of SYT of shape λ is

$$\frac{n!}{\prod h_{i,j}}.$$

$h_{i,j}$



$$\frac{5!}{4\cdot 3\cdot 2\cdot 1\cdot 1} = 5$$

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Let $F(\lambda) = \frac{n!}{\prod h_{i,j}}$ and $G(\lambda)$ the number of SYT of shape λ, so our goal is to show $F(\lambda) = G(\lambda)$.

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$$G := G(\lambda_1, \ldots, \lambda_k) = \sum_{\alpha} G(\lambda_1, \ldots, \lambda_{\alpha-1}, \lambda_\alpha - 1, \lambda_{\alpha+1}, \ldots, \lambda_k) := \sum_{\alpha} G_\alpha.$$

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Thus we'll have $F = G$ if F satisfies this same recurrence relation, or equivalently if

$$1 = \sum_{\alpha} \frac{F_\alpha}{F}.$$

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$$\frac{F_\alpha}{F} = \frac{1}{n} \prod_{1 \le i < \alpha} \frac{h_{i,\beta}}{h_{i,\beta} - 1} \prod_{1 \le j < \beta} \frac{h_{\alpha,j}}{h_{\alpha,j} - 1}$$

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which is exactly the probability that Edmond escapes through (α, β)! □

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A set of vertices $I \subseteq V(G)$ where no two vertices are adjacent is called an *independent set*, so the above problem is really asking to find the largest independent set in G, which we denote by $\alpha(G)$. Doing this in general is a hard problem, but still one can ask for reasonable bounds in terms of parameters of G.

Turán's Theorem

Theorem (Caro-Wei Bound)

Let G be an n-vertex graph with degrees $d_1, \ldots, d_n$. Then

$$\alpha(G) \geq \sum \frac{1}{d_i + 1}.$$

Moreover, equality holds if and only if G is a disjoint union of cliques.

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To achieve this, let $\pi = \pi_1 \cdots \pi_n$ be a uniformly random permutation of the vertices of G, and let I consist of all the vertices u such that $\pi_u^{-1} < \pi_v^{-1}$ for every neighbor v of u.

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In particular, there exists a (deterministic) choice of I with size at least $\sum \frac{1}{d_i+1}$, and hence G has an independent set of at least this size. $\square$

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If G is an n-vertex graph which is K_r-free (i.e. which contains no r vertices which are all adjacent). Then

$$e(G) \leq \left\lfloor \binom{r-1}{2} (n/(r-1))^2 \right\rfloor,$$

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For example, if G is a triangle-free graph then it has at most $\lfloor n^2/4 \rfloor$ edges, and equality holds iff G is a complete balanced bipartite graph.

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The End