Eulerian Polynomials of Digraphs

Sam Spiro, Georgia State University

Joint with Kyle Celano and Nicholas Sieger

Permutation Statistics

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A *descent* of a permutation σ on the set $[n] := \{1, 2, \ldots, n\}$ is an index $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$. An *inversion* is a pair of integers (i, j) with $1 \le i < j \le n$ such that $\sigma(i) > \sigma(j)$.

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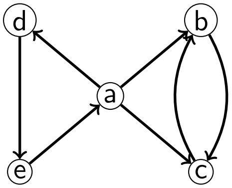
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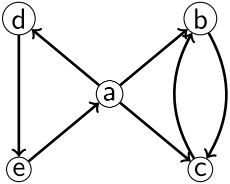

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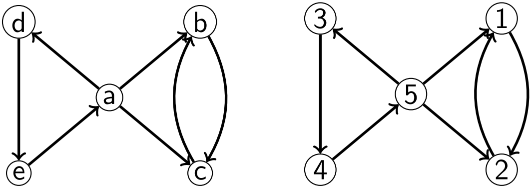

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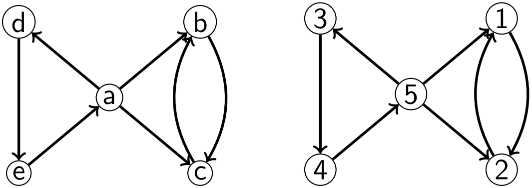

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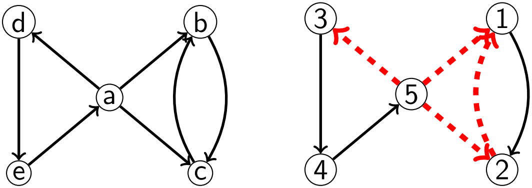

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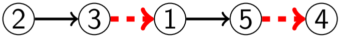

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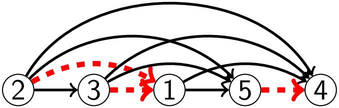

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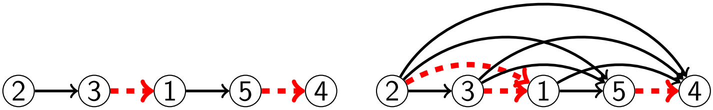

(a) $\mathrm{des}(23154) = 2$ (b) $\mathrm{inv}(23154) = 3$

For a digraph $D = (V, E)$, define

$$A_D(t) = \sum_{\sigma \in \mathfrak{S}_D} t^{\mathrm{des}_D(\sigma)}.$$

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What (if anything) can be said about $A_D(-1)$?

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and the later being the number of *correct proofs of the Riemann hypothesis*[1].

[1] As of the time of writing.

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Question (Kalai 2002)

What can be said about $\nu(G)$?

Definition

Given an n-vertex graph G, we say that an ordering $\pi = (\pi_1, \dots, \pi_n)$ of the vertex set $V(G)$ is an *even sequence* if each of the subgraphs $G[\pi_1, \dots, \pi_i]$ induced by the first i vertices of π have an even number of edges for all $1 \leq i \leq n$.

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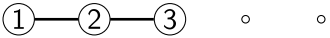

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A permutation π is an even sequence for P_n if and only if π^{-1} is an alternating permutation.

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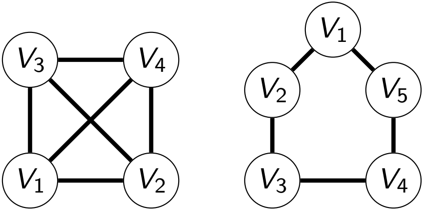
We have $\nu(G) = \eta(G)$ whenever G is bipartite, complete multipartite, or a blowup of a cycle.



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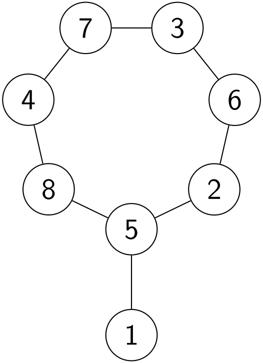

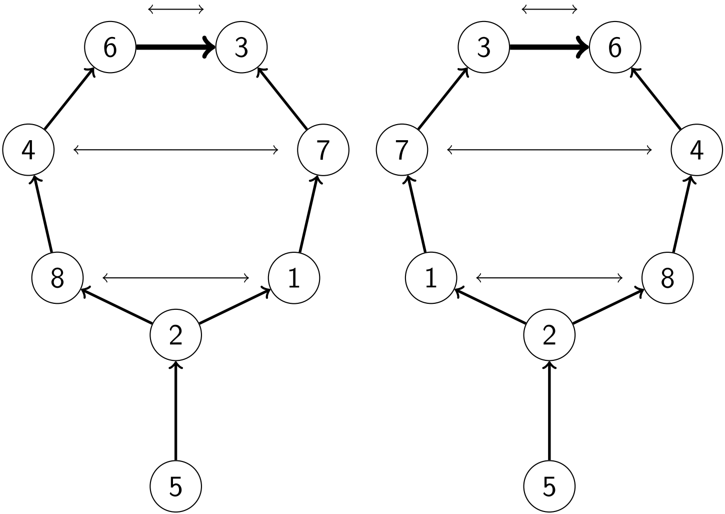
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If G *is a connected graph such that* $\nu(G') = \eta(G')$ *for all induced subgraphs* $G' \subseteq G$*, then* G *is either bipartite, complete multipartite, or a blowup of a cycle.*

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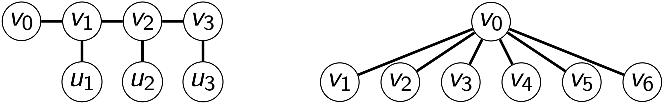
Theorem (Celano, Sieger, S. 2025)

If T is a tree on $2n+1$ vertices, then

$$n!2^n \leq \nu(T) \leq (2n)!$$

Moreover, equality holds in the lower bound if and only if T is a hairbrush, and equality holds in the upper bound if and only if T is a star.



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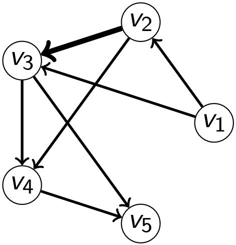

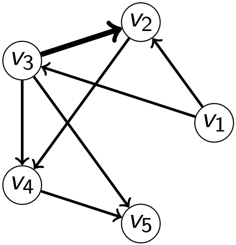

$A_{D_1}(t) = (1+t)^3(1+t+11t^2+t^3+t^4)$ $\quad$ $A_{D_2}(t) = (1+t)(1+5t+16t^2+16t^3+16t^4+5t^5+t^6)$

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Is $\lfloor \frac{n}{2} \rfloor$ the largest $\mathrm{mult}(A_D(t), -1)$ can be?

Note that tournaments have the largest (potential) degree for $A_D(t)$.

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If D is an n-vertex digraph, then

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where $s_2(n)$ denotes the number of 1's in the binary expansion of n.

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where $s_2(n)$ denotes the number of 1's in the binary expansion of n. Moreover, for all n, there exist n-vertex digraphs D with

$$A_D(t) = \frac{n!}{2^{n-s_2(n)}}(1+t)^{n-s_2(n)}.$$

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The only extremal examples we know are "impartial digraphs", which is weird.

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If G has an even number of edges, then

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Corollary

If every bipartite graph with an even number of edges satisfies $\nu(G) = \sum \nu(G - v)$, then $\nu(G) = \eta(G)$ for every bipartite graph.

Lemma

For any digraph,

$$A_D(t) = \sum_{v \in V} \frac{t^{\deg_D^+(v)} + t^{\deg_D^-(v)}}{2} A_{D-v}(t)$$

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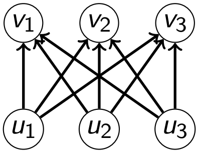

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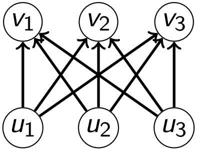

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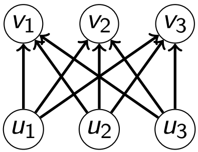

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$$A_D(-1) = \sum_{u \in U} \frac{(-1)^{\deg^+(u)} + 1}{2} A_{D-u}(-1) + \sum_{v \in V} \frac{1 + (-1)^{\deg^-(v)}}{2} A_{D-v}(-1).$$

This same proof implies that if D is a "natural" orientation of a bipartite graph with an even number of edges then

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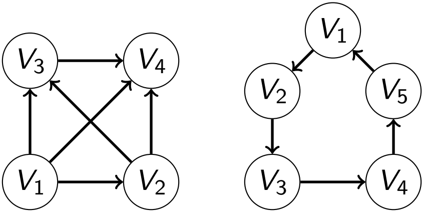
Corollary

$\nu(G) = \eta(G)$ whenever G is bipartite.

Claim

There exists "natural" orientations for complete multipartite graphs/blowup of a cycles which makes it easy to predict the sign of $A_D(t)$.



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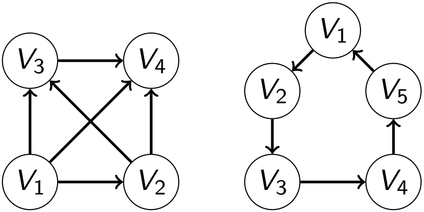

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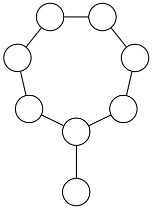

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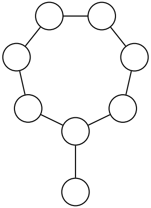

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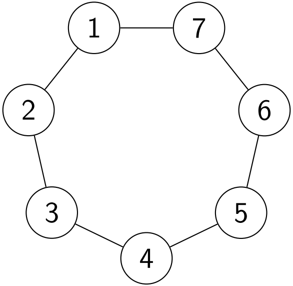
Let G be a connected graph which is induced odd pan-free. Consider a shortest odd cycle.



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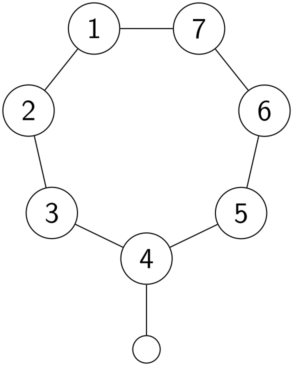

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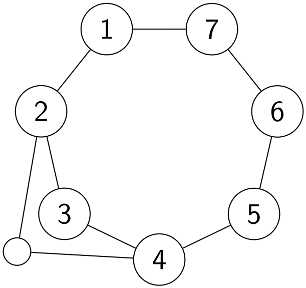

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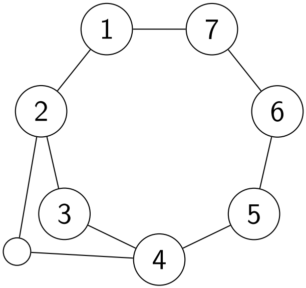

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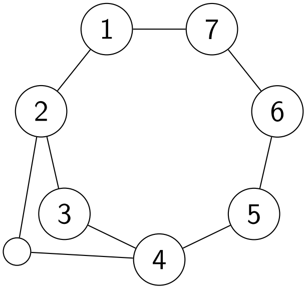

This argument works if G does not have triangles, and if it does we consider a largest clique and play the same game.

Open Problems

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Can one give a combinatorial interpretation for $\nu(G)$ for all graphs G?

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Conjecture

If G is an Eulerian graph, then $\nu(G) = \sum_v \nu(G - v)$.

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Problem

For any bipartite graph $G = ([n], E)$ and orientation D of G, construct an explicit involution $\phi : \mathfrak{S}_n \to \mathfrak{S}_n$ such that

(a) The set of fixed points $\mathcal{F}_\phi$ of ϕ is the set of (inverses of) even sequences of G, and

(b) $(-1)^{\mathrm{des}_D(\sigma)} = -(-1)^{\mathrm{des}_D(\phi(\sigma))}$ for all $\sigma \notin \mathcal{F}_\phi$.

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If D is a tournament on n vertices, then $\mathrm{mult}(A_D(t), -1) = \lfloor \frac{n}{2} \rfloor$.

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Question

Does there exist a digraph D such that $A_D(t)$ has an integral root which is not equal to either 0 or -1?

No such digraph exists on at most 5 vertices, and there exist digraphs with real roots of magnitude larger than 2.

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Question

What can be said about other -1 evaluations of other variants of Eulerian numbers/polynomials?