Forbidden Configurations and Forbidden Families

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Motivation: Extremal Graph Theory

Definition

For a graph G, let $ex(m, G)$ denote the most number of edges a graph on m vertices can have before containing a subgraph isomorphic to G.

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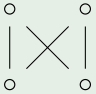

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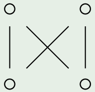

$$ex(m, P_2) = \lfloor m/2 \rfloor$$

$$ex(m, K_{2,2}) = \Theta(m^{3/2})$$

Motivation: Extremal Graph Theory

Theorem (Erdős-Stone)

Let $r = \chi(G)$. *Then*

$$ex(m; G) = \left(\frac{r-2}{r-1} + o(1)\right)\binom{n}{2}$$

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$$ex(m; G) = \left(\frac{r-2}{r-1} + o(1)\right)\binom{n}{2}$$

Question

How do we define the extremal number of a hypergraph?

Terminology

Definition (Simple Matrix)

A matrix A is **simple** if A is a $(0, 1)-$matrix with no repeated columns. That is, A is the incidence matrix of a simple hypergraph.

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Definition (Configuration)

For two matrices F and A, we say that F is a **configuration** in A, and write $F \prec A$ if F is a submatrix of A after permuting the rows and columns of A.
We say A has no configuration F, and write $F \nprec A$, if F is not a configuration in A.

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Example

Let $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then $F \prec A$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{col_2, col_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{row_1, row_2} \left[\begin{array}{|cc|c} \hline 0 & 1 & 0 \\ 1 & 1 & 0 \\ \hline \end{array} \right].$$

Avoid and Forb

Question

For a fixed configuration F, how "large" can a simple matrix A be if $F \nprec A$?

Avoid and Forb

Definition (Avoid(m,F))

A matrix A is in the set Avoid(m, F) if:

1. A has m rows.
2. A is a simple matrix.
3. $F \nprec A$.

Avoid and Forb

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Definition (forb(m,F))

Let A be a matrix and let $|A|$ denote the *number of columns of* A. Let F be a $(0, 1)-$matrix. We define

$$\text{forb}(m, F) := \max_{A}\{|A| \mid A \in \text{Avoid}(m, F)\}.$$

A Simple Example: [1]

Definition

forb(m, F) :=how many columns an m-rowed simple matrix avoiding F can have.

Example

Let $F = [1]$. Then, forb$(m, [1]) = 1$, for all $m \geq 1$.

$$A = \begin{bmatrix} 0 & ? \\ 0 & ? \\ \vdots & \vdots \end{bmatrix}$$

A Less Simple Example: $1_{2,2}$

Theorem

Let $1_{2,2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. *Then,* $\mathit{forb}(m, 1_{2,2}) = 1 + m + \binom{m}{2}$.

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Theorem

Let $1_{2,2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$*. Then,* $forb(m, 1_{2,2}) = 1 + m + \binom{m}{2}$*.*

For the lower bound, take A containing the 0-column, all 1-columns, and all 2-columns.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 1 & 1 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad |A| = 1 + m + \binom{m}{2}.$$

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For the upper bound, let $A \in \text{Avoid}(m, 1_{2,2})$.

Define A' by taking columns of A with more than two 1's and changing 1's to 0's until the columns all have at most two 1's.

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Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; A' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Note that A' is not simple and $1_{2,2} \prec A'$.

A Less Simple Example: $1_{2,2}$

Claim.

If $A \in \text{Avoid}(m, 1_{2,2})$ then $A' \in \text{Avoid}(m, 1_{2,2})$. That is, A' is simple and $1_{2,2} \nprec A'$.

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Claim.

If $A \in \text{Avoid}(m, 1_{2,2})$ then $A' \in \text{Avoid}(m, 1_{2,2})$. That is, A' is simple and $1_{2,2} \not\prec A'$.

Proof (Claim).

Clearly if $1_{2,2} \not\prec A$ then $1_{2,2} \not\prec A'$ (we get to A' by removing 1's of A). If A' has repeated columns then they must be repeated 2-columns, but repeated 2-columns induce a $1_{2,2}$, and we've already shown that $1_{2,2} \not\prec A'$. Thus A' must be simple. $\square$

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Proof of Theorem.

If $A \in \text{Avoid}(m, 1_{2,2})$ then $|A| = |A'| \leq 1 + m + \binom{m}{2}$. □

The Product Operation

Definition

Given two simple matrices A and B with m_1 rows and m_2 rows respectively, we define their *product* $A \times B$ to be the simple matrix on $m_1 + m_2$ rows whose columns in the first m_1 rows are columns of A and in the bottom m_2 rows are columns of B, and $A \times B$ contains all such columns, i.e. $|A \times B| = |A||B|$.

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Example

$$T_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \; I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T_2 \times I_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \; I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The Product Operation

The product construction is a very useful way to construct large avoiding matrices. In fact, all known asymptotic lower bounds forb(m, F) can be obtained by taking repeated products of the matrices I_m, I_m^c and T_m.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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Conjecture ("Erdős-Stone")

For a configuration F let $X(F)$ denote the largest p so that there exists p matrices A_i equal to either $I_{m/p}, I_{m/p}^c$ or $T_{m/p}$ such that $F \nprec A_1 \times \cdots A_p$. Then forb$(m, F) = \Theta(m^{X(F)})$.

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For a configuration F let $X(F)$ denote the largest p so that there exists p matrices A_i equal to either $I_{m/p}, I^c_{m/p}$ or $T_{m/p}$ such that $F \not\prec A_1 \times \cdots A_p$. Then forb$(m, F) = \Theta(m^{X(F)})$.

Example

For $1_{2,2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, note that for large enough m we have $1_{2,2} \prec T_m$ and $1_{2,2} \prec I^c_m$, so $X(F)$ can only be obtained by taking products of I. $1_{2,2} \not\prec I_{m/2} \times I_{m/2}$ (each column has only two 1's), but $1_{2,2} \prec I_{m/3} \times I_{m/3} \times I_{m/3}$. Thus the conjecture predicts that forb$(m, 1_{2,2})) = \Theta(m^2)$, which is indeed true.

Our Research: Forbidden Families

Definition

Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of $(0,1)-$matrices. Let A be a simple matrix. We say $A \in \text{Avoid}(m, \mathcal{F})$, if $F_i \nprec A$ for all $i \in \{1, \ldots, n\}$ and we define
$\text{forb}(m, \mathcal{F}) := \max_A\{|A| \mid A \in \text{Avoid}(m, \mathcal{F})\}$.

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Example

$$\text{forb}(m, I_2) = \Theta(m)$$
$$\text{forb}(m, T_2) = \Theta(m)$$
$$\text{forb}(m, \{I_2, T_2\}) = 2.$$

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Theorem (Balogh-Bollobás)

$forb(m, I_k, T_k, I_k^c) = O_k(1).$

Our Research: Forbidden Families

Our research looked at certain pairs of “minimal” configurations.

Our Research: Forbidden Families

	$1_{4,1}$	F_9	F_{10}	F_{11}	F_{12}	F_{13}	$0_{4,1}$	F_9^c	F_{10}^c	F_{12}^c
$1_{3,1}$	$\Theta(m^2)$ Rm 2.1	$m+2$ Cr 6.16	$\Theta(1)$ Cr 5.1	$\Theta(m^{3/2})$ Cr 5.3	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(1)$ Cr 5.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
$1_{2,2}$	$\Theta(m^2)$ Rm 2.1	$m+3$ Cr 6.16	$\Theta(1)$ Cr 5.1	$\Theta(m^{3/2})$ Cr 5.5	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(1)$ Cr 5.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
I_3	$\Theta(1)$ Cr 5.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
Q_3	$\Theta(m)$ Cr 4.2	$\Theta(m)$ Th 6.1	$\Theta(m)$ Cr 4.2	$\Theta(m^{3/2})$ Cr 4.13	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m)$ Cr 4.2	$\Theta(m)$ Th 6.1	$\Theta(m)$ Cr 4.2	$\Theta(m^2)$ Rm 2.1
Q_8	$\Theta(m)$ Pr 3.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m)$ Pr 3.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
Q_9	$3m-2$ Cr 7.3	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$3m-2$ Cr 7.3	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1	$\Theta(m^2)$ Rm 2.1
$1_{4,1}$		$m+5$ Cr 6.16	$\Theta(1)$ Cr 5.1	$\Theta(m^{3/2})$ Pr 5.7	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.3	$\Theta(1)$ Cr 5.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1
F_9			$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.4	$\Theta(m^2)$ Pr 3.4	$\Theta(m^3)$ Rm 2.1
F_{10}				$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1	$\Theta(m^2)$ Pr 3.4	$\Theta(m^2)$ Pr 3.4	$\Theta(m^3)$ Rm 2.1
F_{11}					$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^{3/2})$ Pr 5.7	$\Theta(m^2)$ Pr 3.3	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1
F_{12}						$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1	$\Theta(m^3)$ Rm 2.1
F_{13}							$\Theta(m^2)$ Pr 3.3	$\Theta(m^2)$ Pr 3.3	$\Theta(m^2)$ Pr 3.3	$\Theta(m^3)$ Rm 2.1

$1_{2,2}$ and F_{11}

Let $1_{2,2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $F_{11} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

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First, observe that $F_{11} = A(K_{2,2})$. That is, F_{11} is the incidence matrix of $K_{2,2}$. It is known that $\text{forb}(m, F_{11}) = \Theta(n^3)$.

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Question

What is $\text{forb}(m, \{1_{2,2}, F_{11}\})$?

$1_{2,2}$ and F_{11}

Let $A \in \text{Avoid}(m, \{1_{2,2}, F_{11}\})$ and consider A' (A after downgrading columns to have two or fewer 1's).

We already know that A' is simple and that $1_{2,2} \nprec A'$, but are we guaranteed that $F_{11} \nprec A'$?

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We already know that A' is simple and that $1_{2,2} \nprec A'$, but are we guaranteed that $F_{11} \nprec A'$?

If $F_{11} \prec A'$ then that means that $\hat{F}_{11} \prec A$, where $\hat{F}_{11}$ denotes F_{11} with some number of 0's changed to 1's.

$1_{2,2}$ and F_{11}

Claim.

If $\hat{F}_{11}$ denotes F_{11} with some number of 0's changed to 1's then $1_{2,2} \prec \hat{F}_{11}$

Example

$$F_{11} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} . \ \hat{F}_{11} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & \mathbf{1}' & 1 & \mathbf{1} \\ 1 & 0 & 1 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} \end{bmatrix} , \begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ \mathbf{1}' & \mathbf{1} & 0 & 1 \end{bmatrix}$$

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But if $1_{2,2} \prec \hat{F}_{11}$ and $1_{2,2} \not\prec A$, then we can't have $\hat{F}_{11} \prec A$. Thus $A' \in \text{Avoid}(m, \{1_{2,2}, F_{11}\})$.

$1_{2,2}$ and F_{11}

We've now reduced the problem to computing how large $|A'|$ can be for $A' \in \text{Avoid}(m, \{1_{2,2}, F_{11}\})$ with A' having only 0, 1 or 2-columns.

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As each column of $1_{2,2}$ and F_{11} has more than one 1, A' can contain all 0 and 1-columns.

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As each column of $1_{2,2}$ and F_{11} has more than one 1, A' can contain all 0 and 1-columns. It is also clear that having any number of 2-columns can't induce $1_{2,2}$, so all we have to figure out is how many 2-columns an m-rowed matrix have before containing F_{11}.

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We can view an m-rowed simple matrix with only 2-columns as the incidence matrix of a graph with m vertices. The condition of avoiding F_{11} as a configuration is equivalent to avoiding $K_{2,2}$ as a subgraph (since $F_{11} = A(K_{2,2})$).

$1_{2,2}$ and F_{11}

Theorem (Sali-S. 2017)

$$forb(m, \{1_{2,2}, F_{11}\}) = 1 + m + ex(m, K_{2,2}) = \Theta(m^{3/2}).$$

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Remark

$$\frac{3}{2} \notin \mathbb{Z}.$$

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$$forb(m, \{1_{2,2}, A(K_{r,s})\}) = 1 + m + ex(m, K_{r,s}).$$

Remark

$$\frac{3}{2} \notin \mathbb{Z}.$$

In particular, the "Erdős-Stone" conjecture doesn't generalize to forbidden families.

Q_3 and F_{11}

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

I claim that $\text{forb}(m, Q_3) = \Theta(m^2)$ and $I \times I^c$ is the only product construction giving this lower bound.

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Theorem (Sali-S. 2017)

All "large" matrices in Avoid(m, Q_3) "look like" $I \times I^c$.

Q_3 and F_{11}

Theorem (Sali-S. 2017)

Let $A \in Avoid(m, Q_3)$ with $|A| = \omega(m \log m)$. There exists a set of integers $\{k_1, \dots, k_y\}$ and a set $A' = \{A'_1, \dots, A'_y\}$, of disjoint submatrices $A'_j \prec A$ such that:

1. $k_{j+1} \le \frac{1}{2} k_j$ *for all j, and $y \le \log m$.*
2. *There exists k_j rows of A such that the columns of A'_j restricted to these rows are columns of I_{k_j}.*
3. *If i is a column of I_{k_j}, let C_i^j denote the set of columns of A'_j that are equal to i when restricted to the k_j rows mentioned above. Then, besides these k_j rows, no row restricted to C_i^j is sparse, and every column of C_i^j is identified by some dense row.*
4. $|A| = \Theta(\sum |A'_j|)$.

Q_3 and F_{11}

Question

What is $\text{forb}(m, Q_3, I_r \times I_s^c)$?

Q_3 and F_{11}

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Theorem (Sali-S. 2017)

If $s \leq r$, then

- $\text{forb}(m, Q_3, I_r \times I_s^c) = O(m^{2-1/s})$.

Q_3 and F_{11}

Question

What is forb$(m, Q_3, I_r \times I_s^c)$?

Theorem (Sali-S. 2017)

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- $forb(m, Q_3, I_r \times I_s^c) = O(m^{2-1/s})$.
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Corollary (Sali-S. 2017)

$\mathit{forb}(m, Q_3, F_{11}) = \mathit{forb}(m, Q_3, I_2 \times I_2^c) = \Theta(m^{3/2})$.

Open Question

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What is $\text{forb}(m, \{1_{2,3}, F_{11}\})$, where $1_{2,3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$?

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More generally, what is $\text{forb}(m, \{1_{2,\ell}, A(K_{r,s})\})$?

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More generally, what is forb$(m, \{1_{2,\ell}, A(K_{r,s})\})$?

Theorem (Sali-S.)

$$forb(m, \{1_{2,\ell}, A(K_{r,s})\}) = \Omega(ex(m, K_{r,s}))$$

$$forb(m, \{1_{2,\ell}, A(K_{r,s})\}) = O(ex(m, K_{r+(\ell-1)\binom{s}{2}, s+(\ell-1)\binom{r}{2}}))$$

The End

Thank You!