New Eigenvalue Bound for the Fractional Chromatic Number

Sam Spiro, Rutgers University

Joint with Krystal Guo

Eigenvalues and Chromatic Numbers

Eigenvalues and Chromatic Numbers

Throughout we let λ_i denote the ith largest eigenvalue of the adjacency matrix of G.

Eigenvalues and Chromatic Numbers

Throughout we let λ_i denote the ith largest eigenvalue of the adjacency matrix of G.

Theorem (Wilf 1967)

$$\chi(G) \leq 1 + \lambda_1.$$

Eigenvalues and Chromatic Numbers

Throughout we let λ_i denote the ith largest eigenvalue of the adjacency matrix of G.

Theorem (Wilf 1967)

$$\chi(G) \leq 1 + \lambda_1.$$

Theorem (Hoffman 1970)

$$\chi(G) \geq 1 + \frac{\lambda_1}{|\lambda_n|}.$$

Eigenvalues and Chromatic Numbers

Elphick and Wocjan defined

$$s^+ = s^+(G) := \sum_{\lambda_i > 0} \lambda_i^2, \qquad s^- = s^-(G) := \sum_{\lambda_i < 0} \lambda_i^2.$$

Eigenvalues and Chromatic Numbers

Elphick and Wocjan defined

$$s^+ = s^+(G) := \sum_{\lambda_i > 0} \lambda_i^2, \qquad s^- = s^-(G) := \sum_{\lambda_i < 0} \lambda_i^2.$$

Theorem (Ando-Lin 2015)

$$\chi(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Eigenvalues and Chromatic Numbers

Elphick and Wocjan defined

$$s^+ = s^+(G) := \sum_{\lambda_i > 0} \lambda_i^2, \qquad s^- = s^-(G) := \sum_{\lambda_i < 0} \lambda_i^2.$$

Theorem (Ando-Lin 2015)

$$\chi(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Sharp for bipartite graphs, cliques K_n, Payley graph on 9 vertices,...

Eigenvalues and Chromatic Numbers

We now know

$$\chi(G) \geq 1 + \max\left\{\frac{\lambda_1}{|\lambda_n|}, \frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Eigenvalues and Chromatic Numbers

We now know

$$\chi(G) \geq 1 + \max\left\{\frac{\lambda_1}{|\lambda_n|}, \frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

These bounds are incomparable with each other.

Eigenvalues and Chromatic Numbers

We now know

$$\chi(G) \geq 1 + \max\left\{\frac{\lambda_1}{|\lambda_n|}, \frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

These bounds are incomparable with each other. In particular, of the 11,855 graphs on 5,6,7,8 vertices which are connected and non-bipartite, Ando-Lin does better than Hoffman for 11,014 of them.

Eigenvalues and Chromatic Numbers

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Eigenvalues and Chromatic Numbers

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Corollary

True if G is d-regular with $d \geq 3$.

Eigenvalues and Chromatic Numbers

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Corollary

True if G is d-regular with $d \geq 3$.

This holds for $G = K_{d+1}$.

Eigenvalues and Chromatic Numbers

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Corollary

True if G is d-regular with $d \geq 3$.

This holds for $G = K_{d+1}$. Otherwise, by Brook's Theorem,

$$d \geq \chi(G)$$

Eigenvalues and Chromatic Numbers

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Corollary

True if G is d-regular with $d \geq 3$.

This holds for $G = K_{d+1}$. Otherwise, by Brook's Theorem,

$$d \geq \chi(G) \geq 1 + \frac{s^+}{s^-}$$

Eigenvalues and Chromatic Numbers

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Corollary

True if G is d-regular with $d \geq 3$.

This holds for $G = K_{d+1}$. Otherwise, by Brook's Theorem,

$$d \geq \chi(G) \geq 1 + \frac{s^+}{s^-} = \frac{s^- + s^+}{s^-}$$

Eigenvalues and Chromatic Numbers

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Corollary

True if G is d-regular with $d \geq 3$.

This holds for $G = K_{d+1}$. Otherwise, by Brook's Theorem,

$$d \geq \chi(G) \geq 1 + \frac{s^+}{s^-} = \frac{s^- + s^+}{s^-} = \frac{dn}{s^-}.$$

Eigenvalues and Chromatic Numbers

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Corollary

True if G is d-regular with $d \geq 3$.

This holds for $G = K_{d+1}$. Otherwise, by Brook's Theorem,

$$d \geq \chi(G) \geq 1 + \frac{s^+}{s^-} = \frac{s^- + s^+}{s^-} = \frac{dn}{s^-}.$$

So $s^- \geq n$, and a similar argument works for s^+. □

Eigenvalues and Chromatic Numbers

Theorem (Ando-Lin 2015)

$$\chi(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Eigenvalues and Chromatic Numbers

Theorem (Ando-Lin 2015)

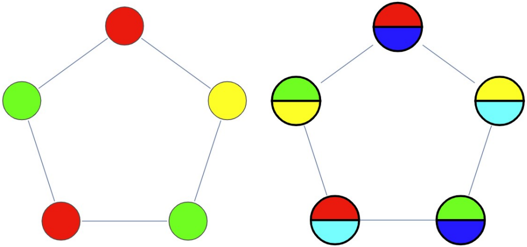
$$\chi(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Theorem (Guo-S. 2022)

$$\chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Fractional Chromatic Number

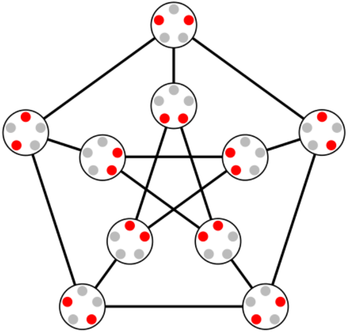
Fractional Chromatic Number



Picture from Wolfram Alpha.

Fractional Chromatic Number

The *Kneser graph* $K_{v;n}$ is the graph whose vertex set consists of n-element subsets of $[v]$ where two sets are adjacent if they are disjoint.



Picture from Wikipedia.

Fractional Chromatic Number

Recall that a map $\phi : V(G) \to V(H)$ is a *homomorphism* if $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(H)$

Fractional Chromatic Number

Recall that a map $\phi : V(G) \to V(H)$ is a *homomorphism* if $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(H)$, and given G we let Φ be the set of graphs H which G has a homomorphism to.

Fractional Chromatic Number

Recall that a map $\phi : V(G) \to V(H)$ is a *homomorphism* if $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(H)$, and given G we let Φ be the set of graphs H which G has a homomorphism to.

Theorem

$$\chi_f(G) = \min_{(v,n):K_{v;n}\in\Phi} \frac{v}{n}.$$

Fractional Chromatic Number

Recall that a map $\phi : V(G) \to V(H)$ is a *homomorphism* if $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(H)$, and given G we let Φ be the set of graphs H which G has a homomorphism to.

Theorem

$$\chi_f(G) = \min_{(v,n):K_{v;n}\in\Phi} \frac{v}{n}.$$

This is analogous to

$$\chi(G) = \min_{K_r\in\Phi} r,$$

since an r-coloring of G is equivalent to a homomorphism $\phi : V(G) \to V(K_r)$.

Fractional Chromatic Number

Theorem (Guo-S. 2022)

$$\chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Fractional Chromatic Number

Theorem (Guo-S. 2022)

$$\chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Theorem (Guo-S. 2022)

If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Fractional Chromatic Number

Theorem (Guo-S. 2022)

$$\chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Theorem (Guo-S. 2022)

If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

This implies the previous result since

$$\chi_f(G) = \min_{(v,n):K_{v;n}\in\Phi} \frac{v}{n} = 1 + \min_{(v,n):K_{v;n}\in\Phi} \frac{\lambda_{\max}(K_{v;n})}{|\lambda_{\min}(K_{v;n})|}.$$

Applications

Theorem (Guo-S. 2022)

$$\chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} = \frac{2e(G)}{\min\{s^+, s^-\}}.$$

Applications

Theorem (Guo-S. 2022)

$$\chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} = \frac{2e(G)}{\min\{s^+, s^-\}}.$$

Corollary (Abiad-De Lima-Desai-Guo-Hogben-Madrid)

The graph $G = C_n$ *satisfies* $\min\{s^+, s^-\} \geq n - 1$.

Applications

Theorem (Guo-S. 2022)

$$\chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} = \frac{2e(G)}{\min\{s^+, s^-\}}.$$

Corollary (Abiad-De Lima-Desai-Guo-Hogben-Madrid)

The graph $G = C_n$ *satisfies* $\min\{s^+, s^-\} \geq n - 1$.

If n is even then $s^+ = s^- = n$

Applications

Theorem (Guo-S. 2022)

$$\chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} = \frac{2e(G)}{\min\{s^+, s^-\}}.$$

Corollary (Abiad-De Lima-Desai-Guo-Hogben-Madrid)

The graph $G = C_n$ *satisfies* $\min\{s^+, s^-\} \geq n - 1$*.*

If n is even then $s^+ = s^- = n$, otherwise

$$\frac{2n}{s^\pm} \leq \chi_f(G) = \frac{2n}{n-1}.$$

Applications

Theorem (Guo-S. 2022)

If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Applications

Theorem (Guo-S. 2022)

If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Corollary (ADDGHM)

If G is a connected unicyclic graph with cycle length $m \gg \sqrt{n}$, then $\min\{s^+, s^-\} \geq n - 1$.

Applications

Theorem (Guo-S. 2022)

If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Applications

Theorem (Guo-S. 2022)

If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

One can extend this result somewhat

Applications

Theorem (Guo-S. 2022)

If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

One can extend this result somewhat, but it does not hold with "edge-transitive" replaced by "vertex-transitive"

Applications

Theorem (Guo-S. 2022)

If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

One can extend this result somewhat, but it does not hold with "edge-transitive" replaced by "vertex-transitive" (e.g. it fails for $G = K_3$ and $H = \overline{C_6}$).

Proof

Proof

Throughout we use the Frobenius norm

$$\|X\|^2 = \sum X_{i,j}^2 \left(= \operatorname{Tr}(X^2) = \sum \lambda_i^2 \right).$$

Proof

Throughout we use the Frobenius norm

$$\|X\|^2 = \sum X_{i,j}^2 \left(= \mathrm{Tr}(X^2) = \sum \lambda_i^2 \right).$$

Observe that if $A = \sum \lambda_i u_i u_i^T$, then we can write $A = X - Y$ with

$$X = \sum_{i:\lambda_i > 0} \lambda_i u_i u_i^T, \; Y = -\sum_{i:\lambda_i < 0} \lambda_i u_i u_i^T.$$

Proof

Throughout we use the Frobenius norm

$$\|X\|^2 = \sum X_{i,j}^2 \left(= \mathrm{Tr}(X^2) = \sum \lambda_i^2 \right).$$

Observe that if $A = \sum \lambda_i u_i u_i^T$, then we can write $A = X - Y$ with

$$X = \sum_{i:\lambda_i > 0} \lambda_i u_i u_i^T, \; Y = -\sum_{i:\lambda_i < 0} \lambda_i u_i u_i^T.$$

With this X, Y are PSD and $\|X\|^2 = s^+, \; \|Y\|^2 = s^-$.

Proof

Throughout we use the Frobenius norm

$$\|X\|^2 = \sum X_{i,j}^2 \left(= \mathrm{Tr}(X^2) = \sum \lambda_i^2\right).$$

Observe that if $A = \sum \lambda_i u_i u_i^T$, then we can write $A = X - Y$ with

$$X = \sum_{i:\lambda_i > 0} \lambda_i u_i u_i^T, \; Y = -\sum_{i:\lambda_i < 0} \lambda_i u_i u_i^T.$$

With this X, Y are PSD and $\|X\|^2 = s^+, \; \|Y\|^2 = s^-$. Thus to prove $s^+/s^- \leq r$, it suffices to prove a bound of the form

$$\|X\|^2 \leq r \|Y\|^2.$$

Proof

Say $\phi : V(G) \to V(H)$ is a homomorphism.

Proof

Say $\phi : V(G) \to V(H)$ is a homomorphism. For $u, v \in V(H)$ and a matrix X indexed by $V(G)$, let $X_{[u,v]}$ be the sumbatrix whose rows and columns are indexed by $\phi^{-1}(u)$ and $\phi^{-1}(v)$ respectively.

Proof

Say $\phi : V(G) \to V(H)$ is a homomorphism. For $u, v \in V(H)$ and a matrix X indexed by $V(G)$, let $X_{[u,v]}$ be the sumbatrix whose rows and columns are indexed by $\phi^{-1}(u)$ and $\phi^{-1}(v)$ respectively.

Lemma (Ando-Lin)

Let $\phi : V(G) \to V(H)$ be a homomorphism, let X, Y be matrices indexed by $V(G)$ such that $XY = 0$ and $X_{[u,v]} = Y_{[u,v]}$ whenever $\{u, v\} \notin E(H)$. If

$$\|X\|^2 \leq (r+1) \sum_{(u,v):\{u,v\}\notin E(H)} \left\|X_{[u,v]}\right\|^2,$$

then

$$\|X\|^2 \leq r\,\|Y\|^2.$$

Proof

Say $\phi : V(G) \to V(H)$ is a homomorphism. For $u, v \in V(H)$ and a matrix X indexed by $V(G)$, let $X_{[u,v]}$ be the sumbatrix whose rows and columns are indexed by $\phi^{-1}(u)$ and $\phi^{-1}(v)$ respectively.

Lemma (Ando-Lin)

Let $\phi : V(G) \to V(H)$ be a homomorphism, let X, Y be matrices indexed by $V(G)$ such that $XY = 0$ and $X_{[u,v]} = Y_{[u,v]}$ whenever $\{u, v\} \notin E(H)$. If

$$\|X\|^2 \leq (r+1) \sum_{(u,v):\{u,v\}\notin E(H)} \left\|X_{[u,v]}\right\|^2,$$

then

$$\|X\|^2 \leq r \|Y\|^2.$$

If $\{u, v\} \notin E(H)$, then $A_{[u,v]} = 0$,

Proof

Say $\phi : V(G) \to V(H)$ is a homomorphism. For $u, v \in V(H)$ and a matrix X indexed by $V(G)$, let $X_{[u,v]}$ be the sumbatrix whose rows and columns are indexed by $\phi^{-1}(u)$ and $\phi^{-1}(v)$ respectively.

Lemma (Ando-Lin)

Let $\phi : V(G) \to V(H)$ be a homomorphism, let X, Y be matrices indexed by $V(G)$ such that $XY = 0$ and $X_{[u,v]} = Y_{[u,v]}$ whenever $\{u, v\} \notin E(H)$. If

$$\|X\|^2 \leq (r+1) \sum_{(u,v):\{u,v\} \notin E(H)} \left\|X_{[u,v]}\right\|^2,$$

then

$$\|X\|^2 \leq r \, \|Y\|^2.$$

If $\{u, v\} \notin E(H)$, then $A_{[u,v]} = 0$, so $A = X - Y$ implies $X_{[u,v]} = Y_{[u,v]}$

Proof

Say $\phi : V(G) \to V(H)$ is a homomorphism. For $u, v \in V(H)$ and a matrix X indexed by $V(G)$, let $X_{[u,v]}$ be the sumbatrix whose rows and columns are indexed by $\phi^{-1}(u)$ and $\phi^{-1}(v)$ respectively.

Lemma (Ando-Lin)

Let $\phi : V(G) \to V(H)$ be a homomorphism, let X, Y be matrices indexed by $V(G)$ such that $XY = 0$ and $X_{[u,v]} = Y_{[u,v]}$ whenever $\{u, v\} \notin E(H)$. If

$$\|X\|^2 \leq (r+1) \sum_{(u,v):\{u,v\}\notin E(H)} \left\|X_{[u,v]}\right\|^2,$$

then

$$\|X\|^2 \leq r\,\|Y\|^2.$$

If $\{u, v\} \notin E(H)$, then $A_{[u,v]} = 0$, so $A = X - Y$ implies $X_{[u,v]} = Y_{[u,v]}$, and the condition $XY = 0$ holds if
$X = \sum_{i:\lambda_i>0} \lambda_i u_i u_i^T$, $Y = -\sum_{i:\lambda_i<0} \lambda_i u_i u_i^T$.

Proof

We've reduced our problem to figuring out when a PSD matrix X satisfies

$$\|X\|^2 \leq (r+1) \sum_{(u,v):\{u,v\} \notin E(H)} \left\|X_{[u,v]}\right\|^2 .$$

Proof

We've reduced our problem to figuring out when a PSD matrix X satisfies

$$\|X\|^2 \leq (r+1) \sum_{(u,v):\{u,v\}\notin E(H)} \left\|X_{[u,v]}\right\|^2 .$$

Lemma (Ando-Lin)

The result above holds if $H = K_{r+1}$.

Proof

Lemma (Ando-Lin)

If X is a PSD matrix with blocks indexed by $\{1, 2, \ldots, r+1\}$, then

$$\|X\|^2 \leq (r+1) \sum_{u \in \{1,2,\ldots,r+1\}} \left\|X_{[u,u]}\right\|^2 .$$

Proof

Lemma (Ando-Lin)

If X is a PSD matrix with blocks indexed by $\{1, 2, \ldots, r+1\}$, then

$$\|X\|^2 \leq (r+1) \sum_{u \in \{1,2,\ldots,r+1\}} \left\|X_{[u,u]}\right\|^2 .$$

X being PSD implies $\begin{bmatrix} X_{[u,u]} & X_{[u,v]} \\ X_{[v,u]} & X_{[v,v]} \end{bmatrix}$ is

Proof

Lemma (Ando-Lin)

If X is a PSD matrix with blocks indexed by $\{1, 2, \ldots, r+1\}$, then

$$\|X\|^2 \leq (r+1) \sum_{u \in \{1,2,\ldots,r+1\}} \left\|X_{[u,u]}\right\|^2 .$$

X being PSD implies $\begin{bmatrix} X_{[u,u]} & X_{[u,v]} \\ X_{[v,u]} & X_{[v,v]} \end{bmatrix}$ is, and hence

$$\left\|X_{[u,v]}\right\|^2 \leq \left\|X_{[u,u]}\right\| \left\|X_{[v,v]}\right\|$$

Proof

Lemma (Ando-Lin)

If X is a PSD matrix with blocks indexed by $\{1, 2, \ldots, r+1\}$, then

$$\|X\|^2 \leq (r+1) \sum_{u \in \{1,2,\ldots,r+1\}} \left\|X_{[u,u]}\right\|^2 .$$

X being PSD implies $\begin{bmatrix} X_{[u,u]} & X_{[u,v]} \\ X_{[v,u]} & X_{[v,v]} \end{bmatrix}$ is, and hence

$$\left\|X_{[u,v]}\right\|^2 \leq \left\|X_{[u,u]}\right\| \left\|X_{[v,v]}\right\| \leq \frac{\left\|X_{[u,u]}\right\|^2 + \left\|X_{[v,v]}\right\|^2}{2}.$$

Proof

Lemma (Ando-Lin)

If X is a PSD matrix with blocks indexed by $\{1, 2, \ldots, r+1\}$, then

$$\|X\|^2 \leq (r+1) \sum_{u \in \{1,2,\ldots,r+1\}} \left\|X_{[u,u]}\right\|^2 .$$

X being PSD implies $\begin{bmatrix} X_{[u,u]} & X_{[u,v]} \\ X_{[v,u]} & X_{[v,v]} \end{bmatrix}$ is, and hence

$$\left\|X_{[u,v]}\right\|^2 \leq \left\|X_{[u,u]}\right\| \left\|X_{[v,v]}\right\| \leq \frac{\left\|X_{[u,u]}\right\|^2 + \left\|X_{[v,v]}\right\|^2}{2}.$$

Thus

$$\sum_{u \neq v} \left\|X_{[u,v]}\right\|^2 \leq \sum_{u \neq v} \frac{\left\|X_{[u,u]}\right\|^2 + \left\|X_{[v,v]}\right\|^2}{2} = r \sum_{u} \left\|X_{[u,u]}\right\|^2 .$$

Proof

Lemma (Ando-Lin)

If X is a PSD matrix with blocks indexed by $\{1, 2, \ldots, r+1\}$, then

$$\|X\|^2 \le (r+1) \sum_{u \in \{1,2,\ldots,r+1\}} \left\|X_{[u,u]}\right\|^2 .$$

X being PSD implies $\begin{bmatrix} X_{[u,u]} & X_{[u,v]} \\ X_{[v,u]} & X_{[v,v]} \end{bmatrix}$ is, and hence

$$\left\|X_{[u,v]}\right\|^2 \le \left\|X_{[u,u]}\right\| \left\|X_{[v,v]}\right\| \le \frac{\left\|X_{[u,u]}\right\|^2 + \left\|X_{[v,v]}\right\|^2}{2}.$$

Thus

$$\sum_{u \ne v} \left\|X_{[u,v]}\right\|^2 \le \sum_{u \ne v} \frac{\left\|X_{[u,u]}\right\|^2 + \left\|X_{[v,v]}\right\|^2}{2} = r \sum_{u} \left\|X_{[u,u]}\right\|^2 .$$

Hence

$$\|X\|^2 = \sum_{u} \left\|X_{[u,u]}\right\|^2 + \sum_{u \ne v} \left\|X_{[u,v]}\right\|^2 \le (r+1) \sum_{u} \left\|X_{[u,u]}\right\|^2 .$$

Proof

Lemma (Main)

If X is PSD and H is vertex and edge-transitive, then

$$\|X\|^2 \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} \left\| X_{[u,v]} \right\|^2 .$$

Proof

Lemma (Main)

If X is PSD and H is vertex and edge-transitive, then

$$\|X\|^2 \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1\right) \sum_{(u,v):\{u,v\}\notin E(H)} \left\|X_{[u,v]}\right\|^2 .$$

To make things simpler, consider the matrix Z indexed by $V(H)$ with $Z_{u,v} := \left\|X_{[u,v]}\right\|^2$.

Proof

Lemma (Main)

If X is PSD and H is vertex and edge-transitive, then

$$\|X\|^2 \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} \left\| X_{[u,v]} \right\|^2 .$$

To make things simpler, consider the matrix Z indexed by $V(H)$ with $Z_{u,v} := \left\| X_{[u,v]} \right\|^2$.

Claim

It suffices to show that for non-negative PSD matrices Z, we have

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v} .$$

Proof

Lemma

If H is edge-transitive, then there exists a set of non-empty graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$ on $V(H)$ satisfying the following properties:

Proof

Lemma

If H is edge-transitive, then there exists a set of non-empty graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$ on $V(H)$ satisfying the following properties:

1. $H = H_n$;

Proof

Lemma

If H is edge-transitive, then there exists a set of non-empty graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$ on $V(H)$ satisfying the following properties:

1. $H = H_n$;
2. *For every pair of distinct vertices $u, v \in V(H)$, there exists a unique i with $uv \in E(H_i)$;*

Proof

Lemma

If H is edge-transitive, then there exists a set of non-empty graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$ on $V(H)$ satisfying the following properties:

1. $H = H_n$;
2. *For every pair of distinct vertices $u, v \in V(H)$, there exists a unique i with $uv \in E(H_i)$;*
3. *Each $\pi \in \mathrm{Aut}(H)$ is an automorphism for each H_i;*

Proof

Lemma

If H is edge-transitive, then there exists a set of non-empty graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$ on $V(H)$ satisfying the following properties:

1. $H = H_n$;
2. *For every pair of distinct vertices $u, v \in V(H)$, there exists a unique i with $uv \in E(H_i)$;*
3. *Each $\pi \in \mathrm{Aut}(H)$ is an automorphism for each H_i; and*
4. *For every pair of edges $uv, xy \in E(H_i)$, there exists $\pi \in \mathrm{Aut}(H)$ with $\pi(uv) = xy$.*

Proof

Lemma

If H is edge-transitive, then there exists a set of non-empty graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$ on $V(H)$ satisfying the following properties:

1. $H = H_n$;
2. *For every pair of distinct vertices $u, v \in V(H)$, there exists a unique i with $uv \in E(H_i)$;*
3. *Each $\pi \in \mathrm{Aut}(H)$ is an automorphism for each H_i; and*
4. *For every pair of edges $uv, xy \in E(H_i)$, there exists $\pi \in \mathrm{Aut}(H)$ with $\pi(uv) = xy$.*

E.g. if $H = K_{v;n}$ then this holds with H_i the graph on $\binom{[v]}{n}$ where $S \sim T$ if $|S \cap T| = n - i$.

Proof

Let A_i be the adjacency matrix of H_i and A_0 the identity matrix.

Proof

Let A_i be the adjacency matrix of H_i and A_0 the identity matrix.

Claim

To show

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1\right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}$$

for Z PSD and non-negative, it suffices to show this when $Z = \sum z_i A_i$.

Proof

Let A_i be the adjacency matrix of H_i and A_0 the identity matrix.

Claim

To show

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}$$

for Z PSD and non-negative, it suffices to show this when $Z = \sum z_i A_i$.

Gizen Z PSD and non-negative, define

$$\overline{Z} = \frac{1}{|\operatorname{Aut}(H)|} \sum_{\pi \in \operatorname{Aut}(H)} P_\pi^T Z P_\pi.$$

Proof

Let A_i be the adjacency matrix of H_i and A_0 the identity matrix.

Claim

To show

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}$$

for Z PSD and non-negative, it suffices to show this when $Z = \sum z_i A_i$.

Gizen Z PSD and non-negative, define

$$\overline{Z} = \frac{1}{|\operatorname{Aut}(H)|} \sum_{\pi \in \operatorname{Aut}(H)} P_\pi^T Z P_\pi .$$

This matrix is PSD and non-negative with $\sum \overline{Z}_{u,v} = \sum Z_{u,v}$, $\sum_{(u,v):\{u,v\} \notin E(H)} \overline{Z}_{u,v} = \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}$, and $\overline{Z} = \sum z_i A_i$. □

Proof

Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}.$$

Proof

Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}.$$

If x is the eigenvector of $A_n = A(H)$ associated to $\lambda_{\min}(A_n)$, then

$$0 \leq \mathrm{x}^T Z \mathrm{x}$$

Proof

Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}.$$

If x is the eigenvector of $A_n = A(H)$ associated to $\lambda_{\min}(A_n)$, then

$$0 \leq \mathrm{x}^T Z \mathrm{x} = \sum_{i=0}^{n} z_i \cdot \mathrm{x}^T A_i \mathrm{x}$$

Proof

Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}.$$

If x is the eigenvector of $A_n = A(H)$ associated to $\lambda_{\min}(A_n)$, then

$$0 \leq \mathrm{x}^T Z \mathrm{x} = \sum_{i=0}^{n} z_i \cdot \mathrm{x}^T A_i \mathrm{x} \leq \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) + z_n \lambda_{\min}(A_n)$$

Proof

Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

$$\sum Z_{u,v} \le \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}.$$

If x is the eigenvector of $A_n = A(H)$ associated to $\lambda_{\min}(A_n)$, then

$$0 \le \mathrm{x}^T Z \mathrm{x} = \sum_{i=0}^{n} z_i \cdot \mathrm{x}^T A_i \mathrm{x} \le \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) + z_n \lambda_{\min}(A_n),$$

i.e.

$$\sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) \ge -z_n \lambda_{\min}(A_n) = z_n |\lambda_{\min}(A_n)|.$$

Proof

Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}.$$

If x is the eigenvector of $A_n = A(H)$ associated to $\lambda_{\min}(A_n)$, then

$$0 \leq \mathrm{x}^T Z \mathrm{x} = \sum_{i=0}^{n} z_i \cdot \mathrm{x}^T A_i \mathrm{x} \leq \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) + z_n \lambda_{\min}(A_n),$$

i.e.

$$\sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) \geq -z_n \lambda_{\min}(A_n) = z_n |\lambda_{\min}(A_n)|.$$

Each of the A_i matrices has $\lambda_{\max}(A_i)|V(H)|$ 1-entries

Proof

Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1\right) \sum_{(u,v):\{u,v\}\notin E(H)} Z_{u,v}.$$

If x is the eigenvector of $A_n = A(H)$ associated to $\lambda_{\min}(A_n)$, then

$$0 \leq \mathrm{x}^T Z \mathrm{x} = \sum_{i=0}^{n} z_i \cdot \mathrm{x}^T A_i \mathrm{x} \leq \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) + z_n \lambda_{\min}(A_n),$$

i.e.

$$\sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) \geq -z_n \lambda_{\min}(A_n) = z_n |\lambda_{\min}(A_n)|.$$

Each of the A_i matrices has $\lambda_{\max}(A_i)|V(H)|$ 1-entries , so the lemma statement is equivalent to saying

$$\sum_{i=0}^{n} z_i \lambda_{\max}(A_i)|V(H)| \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1\right) \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i)|V(H)|$$

Proof

Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{(u,v):\{u,v\} \notin E(H)} Z_{u,v}.$$

If x is the eigenvector of $A_n = A(H)$ associated to $\lambda_{\min}(A_n)$, then

$$0 \leq \mathrm{x}^T Z \mathrm{x} = \sum_{i=0}^{n} z_i \cdot \mathrm{x}^T A_i \mathrm{x} \leq \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) + z_n \lambda_{\min}(A_n),$$

i.e.

$$\sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) \geq -z_n \lambda_{\min}(A_n) = z_n |\lambda_{\min}(A_n)|.$$

Each of the A_i matrices has $\lambda_{\max}(A_i)|V(H)|$ 1-entries , so the lemma statement is equivalent to saying

$$\sum_{i=0}^{n} z_i \lambda_{\max}(A_i)|V(H)| \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i)|V(H)|,$$

and manipulations together with the previous inequality gives the result.

Proof Summary

Proof Summary

1. By writing $A = X - Y$, we reduced showing $s^+ \leq rs^-$ to showing $\|X\|^2 \leq r\|Y\|^2$.

Proof Summary

1. By writing $A = X - Y$, we reduced showing $s^+ \leq rs^-$ to showing $\|X\|^2 \leq r\,\|Y\|^2$.
2. Through some (omitted) arithmetic, we reduced this to proving a certain norm inequality for PSD matrices X.

Proof Summary

1. By writing $A = X - Y$, we reduced showing $s^+ \leq rs^-$ to showing $\|X\|^2 \leq r\|Y\|^2$.
2. Through some (omitted) arithmetic, we reduced this to proving a certain norm inequality for PSD matrices X.
3. By compressing X to Z and using an averaging argument based off of $\mathcal{H}$, we reduced this to an inequality involving PSD $Z = \sum z_i A_i$.

Proof Summary

1. By writing $A = X - Y$, we reduced showing $s^+ \leq rs^-$ to showing $\|X\|^2 \leq r\|Y\|^2$.
2. Through some (omitted) arithmetic, we reduced this to proving a certain norm inequality for PSD matrices X.
3. By compressing X to Z and using an averaging argument based off of $\mathcal{H}$, we reduced this to an inequality involving PSD $Z = \sum z_i A_i$.
4. Finally, we proved this using Rayleigh quotient arguments.

Future Directions

$$\chi(G) \geq \chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Future Directions

$$\chi(G) \geq \chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Question (Anekstein-Elphick-Wocjan)

Is it true that

$$\chi_c(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}$$

where $\chi_c(G)$ is the vector chromatic number?

Future Directions

$$\chi(G) \geq \chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Question (Anekstein-Elphick-Wocjan)

Is it true that

$$\chi_c(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}$$

where $\chi_c(G)$ is the vector chromatic number? Do these bounds hold for orthogonal rank or projective rank?

Future Directions

$$\chi(G) \geq \chi_f(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Question (Anekstein-Elphick-Wocjan)

Is it true that

$$\chi_c(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}$$

where $\chi_c(G)$ is the vector chromatic number? Do these bounds hold for orthogonal rank or projective rank?

Conjecture (Elphick-Wocjan)

$$\chi_f(G) \geq 1 + \max\left\{\frac{n^+}{n^-}, \frac{n^-}{n^+}\right\},$$

where n^+, n^- is the number of positive/negative eigenvalues of G.

Future Directions

Conjecture (Elphick-Wocjan)

If G is a connected n-vertex graph, then

$$\min\{s^+, s^-\} \geq n - 1.$$

Future Directions

Given a family of graphs $\mathcal{H}$, define

$$\chi_{\mathcal{H}}(G) = 1 + \inf_{H \in \mathcal{H} \cap \Phi} \frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|}.$$

Future Directions

Given a family of graphs $\mathcal{H}$, define

$$\chi_{\mathcal{H}}(G) = 1 + \inf_{H \in \mathcal{H} \cap \Phi} \frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|}.$$

Our main result shows for $\mathcal{H}$ the set of edge-transitive graphs that

$$\chi_{\mathcal{H}}(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Future Directions

Given a family of graphs $\mathcal{H}$, define

$$\chi_{\mathcal{H}}(G) = 1 + \inf_{H \in \mathcal{H} \cap \Phi} \frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|}.$$

Our main result shows for $\mathcal{H}$ the set of edge-transitive graphs that

$$\chi_{\mathcal{H}}(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Question

What is the largest set of graphs $\mathcal{H}$ for which the bound above holds?

Future Directions

Given a family of graphs $\mathcal{H}$, define

$$\chi_{\mathcal{H}}(G) = 1 + \inf_{H \in \mathcal{H} \cap \Phi} \frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|}.$$

Our main result shows for $\mathcal{H}$ the set of edge-transitive graphs that

$$\chi_{\mathcal{H}}(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

Question

What is the largest set of graphs $\mathcal{H}$ for which the bound above holds?

Question

What is the largest set of graphs $\mathcal{H}$ such that the bound

$$\chi_{\mathcal{H}}(G) \geq 1 + \frac{\lambda_{\max}(G)}{|\lambda_{\min}(G)|}$$

holds?