New Eigenvalue Bound for the Fractional Chromatic Number

Sam Spiro, Rutgers University

Joint with Krystal Guo

Eigenvalues and Chromatic Numbers

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Theorem (Hoffman 1970)

$$\chi(G) \geq 1 + \frac{\lambda_1}{|\lambda_n|}.$$

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Elphick and Wocjan defined

$$s^+ = s^+(G) := \sum_{\lambda_i > 0} \lambda_i^2, \qquad s^- = s^-(G) := \sum_{\lambda_i < 0} \lambda_i^2.$$

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Sharp for bipartite graphs, cliques K_n, Payley graph on 9 vertices,...

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These bounds are incomparable with each other. In particular, of the 11,855 graphs on 5,6,7,8 vertices which are connected and non-bipartite, Ando-Lin does better than Hoffman for 11,014 of them.

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If G is a connected n-vertex graph, then

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So $s^- \geq n$, and a similar argument works for s^+. □

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Theorem (Coutinho-Spier 2023)

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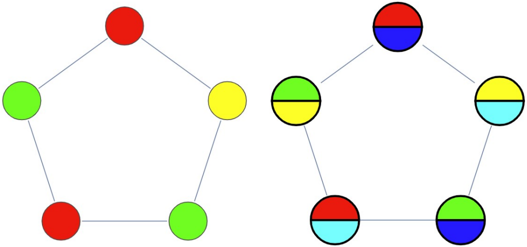
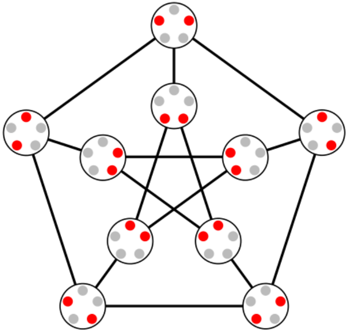


Picture from Wolfram Alpha.

Fractional Chromatic Number

The *Kneser graph* $K_{v;n}$ is the graph whose vertex set consists of n-element subsets of $[v]$ where two sets are adjacent if they are disjoint.



Picture from Wikipedia.

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Theorem

$$\chi_f(G) = \min_{(v,n):K_{v;n}\in\Phi} \frac{v}{n}.$$

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Theorem

$$\chi_f(G) = \min_{(v,n):K_{v;n}\in\Phi} \frac{v}{n}.$$

This is analogous to

$$\chi(G) = \min_{K_r\in\Phi} r,$$

since a proper r-coloring of G is equivalent to a homomorphism $\phi : V(G) \to V(K_r)$.

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If G has a homomorphism to an edge-transitive graph H, then

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

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This implies the previous result since

$$\chi_f(G) = \min_{(v,n):K_{v;n}\in\Phi} \frac{v}{n} = 1 + \min_{(v,n):K_{v;n}\in\Phi} \frac{\lambda_{\max}(K_{v;n})}{|\lambda_{\min}(K_{v;n})|}.$$

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Corollary (Abiad-De Lima-Desai-Guo-Hogben-Madrid 2022)

If G is a connected unicyclic graph with cycle length $m \gg \sqrt{n}$, then $\min\{s^+, s^-\} \geq n - 1$.

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Observe that if $A = \sum \lambda_i u_i u_i^T$, then we can write $A = X - Y$ with

$$X = \sum_{i:\lambda_i > 0} \lambda_i u_i u_i^T, \quad Y = -\sum_{i:\lambda_i < 0} \lambda_i u_i u_i^T.$$

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With this X, Y are PSD and $\|X\|^2 = s^+$, $\|Y\|^2 = s^-$.

Claim

It suffices to show that for non-negative PSD matrices Z,

$$\sum Z_{u,v} \leq \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1\right) \sum_{(u,v):\{u,v\}\notin E(H)} Z_{u,v}.$$

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E.g. if $H = K_{v;n}$ then this holds with H_i the graph on $\binom{[v]}{n}$ where $S \sim T$ if $|S \cap T| = n - i$.

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This matrix is PSD and non-negative with $\sum \overline{Z}_{u,v} = \sum Z_{u,v}$, $\sum_{(u,v):\{u,v\}\notin E(H)} \overline{Z}_{u,v} = \sum_{(u,v):\{u,v\}\notin E(H)} Z_{u,v}$, and $\overline{Z} = \sum z_i A_i$. □

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Lemma

If Z is PSD, non-negative, and $Z = \sum z_i A_i$, then

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If x is the eigenvector of $A_n = A(H)$ associated to $\lambda_{\min}(A_n)$, then

$$0 \le x^T Z x = \sum_{i=0}^{n} z_i \cdot x^T A_i x \le \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) + z_n \lambda_{\min}(A_n),$$

i.e.

$$\sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i) \ge -z_n \lambda_{\min}(A_n) = z_n |\lambda_{\min}(A_n)|.$$

Each of the A_i matrices has $\lambda_{\max}(A_i)|V(H)|$ 1-entries, so the lemma statement is equivalent to saying

$$\sum_{i=0}^{n} z_i \lambda_{\max}(A_i)|V(H)| \le \left(\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} + 1 \right) \sum_{i=0}^{n-1} z_i \lambda_{\max}(A_i)|V(H)|,$$

and manipulations together with the previous inequality gives the result.

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