Generalized Turán Problems for Trees and More

Sam Spiro, Georgia State University

Based on Joint Work with Sean English

Turán Problems

The Turán number $\mathrm{ex}(n, F)$ is defined to be the maximum number of edges that an F-free graph on n vertices can have.

Turán Problems

The Turán number $\mathrm{ex}(n, F)$ is defined to be the maximum number of edges that an F-free graph on n vertices can have.

Theorem

$$\mathrm{ex}(n, K_2) = 0.$$

Turán Problems

The Turán number $\mathrm{ex}(n, F)$ is defined to be the maximum number of edges that an F-free graph on n vertices can have.

Theorem (Spiro's Theorem)

$$\mathrm{ex}(n, K_2) = 0.$$

Turán Problems

Theorem (Turán 1941)

$$\mathrm{ex}(n, K_r) \sim \frac{r-2}{r-1}\binom{n}{2}$$

Turán Problems

Theorem (Turán 1941)

$$\mathrm{ex}(n, K_r) \sim \frac{r-2}{r-1}\binom{n}{2}$$

Theorem (Kővari-Sós-Turán 1954)

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$

Turán Problems

Theorem (Turán 1941)

$$\mathrm{ex}(n, K_r) \sim \frac{r-2}{r-1}\binom{n}{2}$$

Theorem (Kővari-Sós-Turán 1954)

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$

Theorem (Folklore)

If T is a tree with $e(T) \geq 2$, then

$$\mathrm{ex}(n, T) = \Theta(n).$$

Turán Problems

Given a family of graphs $\mathcal{F}$, we say G is $\mathcal{F}$-free if G is F-free for all $F \in \mathcal{F}$. We define $\mathrm{ex}(n, \mathcal{F})$ to be the maximum number of edges in an n-vertex $\mathcal{F}$-free graph.

Turán Problems

Given a family of graphs $\mathcal{F}$, we say G is $\mathcal{F}$-free if G is F-free for all $F \in \mathcal{F}$. We define $\mathrm{ex}(n, \mathcal{F})$ to be the maximum number of edges in an n-vertex $\mathcal{F}$-free graph.

Conjecture (Compactness Conjecture)

If $\mathcal{F}$ is a finite family of graphs which does not include both a star and a matching, then $\mathrm{ex}(n, \mathcal{F}) = \Theta(\mathrm{ex}(n, F))$ for some $F \in \mathcal{F}$.

Generalized Turán Problems

Generalized Turán Problems

Definition (Alon-Shikhelman 2016)

Given a graph H and a family of graphs $\mathcal{F}$, we define the generalized Turán number $\mathrm{ex}(n, H, \mathcal{F})$ to be the maximum number of copies of H in an n-vertex $\mathcal{F}$-free graph.

Generalized Turán Problems

Definition (Alon-Shikhelman 2016)

Given a graph H and a family of graphs $\mathcal{F}$, we define the generalized Turán number $\mathrm{ex}(n, H, \mathcal{F})$ to be the maximum number of copies of H in an n-vertex $\mathcal{F}$-free graph.

Proposition

$$\mathrm{ex}(n, K_2, \mathcal{F}) = \mathrm{ex}(n, \mathcal{F}).$$

Generalized Turán Problems

Theorem (Gerbner-Palmer 2019)

For all $r \geq 2$ and families $\mathcal{F}$ we have

$$\mathrm{ex}(n, K_r, \mathcal{F}) \leq \mathrm{ex}(n, \mathcal{F})^{r/2}.$$

Generalized Turán Problems

Theorem (Gerbner-Palmer 2019)

For all $r \geq 2$ and families $\mathcal{F}$ we have

$$\mathrm{ex}(n, K_r, \mathcal{F}) \leq \mathrm{ex}(n, \mathcal{F})^{r/2}.$$

Theorem (Gerbner 2023)

Determines $\mathrm{ex}(n, K_{2,t}, F)$ asymptotically for any non-bipartite graph F.

Generalized Turán Problems

Theorem (Gerbner-Palmer 2019)

For all $r \geq 2$ and families $\mathcal{F}$ we have

$$\mathrm{ex}(n, K_r, \mathcal{F}) \leq \mathrm{ex}(n, \mathcal{F})^{r/2}.$$

Theorem (Gerbner 2023)

Determines $\mathrm{ex}(n, K_{2,t}, F)$ asymptotically for any non-bipartite graph F.

Theorem (Füredi-Kündgen 2006)

If $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^{2-\beta})$ and $\mathcal{F}$ does not contain a star, then

$$\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \tilde{\Theta}(\max\{n^t, n^{t+1-t\beta}\}).$$

Generalized Turán Problems

Question

Can we prove bounds on $\mathrm{ex}(n, T, \mathcal{F})$ for arbitrary trees T (for some possibly fixed $\mathcal{F}$)?

Generalized Turán Problems

Question

Can we prove bounds on $\mathrm{ex}(n, T, \mathcal{F})$ for arbitrary trees T (for some possibly fixed $\mathcal{F}$)?

Theorem (Gerbner 2023)

Determines the order of magnitude of $\mathrm{ex}(n, T, K_{2,t})$ *for every tree* T.

Generalized Turán Problems

Question

Can we prove bounds on $\mathrm{ex}(n, T, \mathcal{F})$ for arbitrary trees T (for some possibly fixed $\mathcal{F}$)?

Theorem (Gerbner 2023)

Determines the order of magnitude of $\mathrm{ex}(n, T, K_{2,t})$ *for every tree* T.

Theorem (Alon-Shikhelman 2016)

If T *is a tree and* F *is a tree, then* $\mathrm{ex}(n, T, F) = \Theta(n^k)$ *for some integer* k.

Generalized Turán Problems

Question

Can we prove bounds on $\mathrm{ex}(n, T, \mathcal{F})$ for arbitrary trees T (for some possibly fixed $\mathcal{F}$)?

Theorem (Gerbner 2023)

Determines the order of magnitude of $\mathrm{ex}(n, T, K_{2,t})$ *for every tree* T.

Theorem (Alon-Shikhelman 2016)

If T *is a tree and* F *is a tree, then* $\mathrm{ex}(n, T, F) = \Theta(n^k)$ *for some integer* k.

Theorem (Letzter 2019)

If H *is any graph and* F *is a tree, then* $\mathrm{ex}(n, H, F) = \Theta(n^k)$ *for some integer* k.

A General Theorem for Trees

Theorem (English-S. 2025+)

For any tree T, integer $k \geq 0$, and family of graphs $\mathcal{F}$, either

$$\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1}),$$

or

$$\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k).$$

Moreover, we can determine which of these two cases happen for a given $T, k, \mathcal{F}$.

A General Theorem for Trees

Theorem (English-S. 2025+)

Either $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1})$ *or* $\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$.

A General Theorem for Trees

Theorem (English-S. 2025+)

Either $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1})$ *or* $\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$.

Corollary

If $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\epsilon})$, *then for every tree* T *there exists an integer* k *such that*

$$n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+k\epsilon}.$$

A General Theorem for Trees

Theorem (English-S. 2025+)

Either $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1})$ *or* $\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$.

Corollary

If $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\epsilon})$, *then for every tree T there exists an integer k such that*

$$n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+k\epsilon}.$$

Let k be the largest integer such that $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^k)$.

A General Theorem for Trees

Theorem (English-S. 2025+)

Either $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1})$ *or* $\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$.

Corollary

If $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\epsilon})$, *then for every tree* T *there exists an integer* k *such that*

$$n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+k\epsilon}.$$

Let k be the largest integer such that $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^k)$. Because $\mathrm{ex}(n, T, \mathcal{F}) \neq \Omega(n^{k+1})$, our theorem implies

$$\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$$

A General Theorem for Trees

Theorem (English-S. 2025+)

Either $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1})$ *or* $\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$.

Corollary

If $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\epsilon})$*, then for every tree* T *there exists an integer* k *such that*

$$n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+k\epsilon}.$$

Let k be the largest integer such that $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^k)$. Because $\mathrm{ex}(n, T, \mathcal{F}) \neq \Omega(n^{k+1})$, our theorem implies

$$\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k) = O(n^{k+k\epsilon}).$$

A General Theorem for Trees

Theorem (English-S. 2025+)

Either $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1})$ *or* $\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$.

Corollary

If $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\epsilon})$, *then for every tree* T *there exists an integer* k *such that*

$$n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+k\epsilon}.$$

Let k be the largest integer such that $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^k)$. Because $\mathrm{ex}(n, T, \mathcal{F}) \neq \Omega(n^{k+1})$, our theorem implies

$$\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k) = O(n^{k+k\epsilon}).$$

Corollary

If T *is a tree and* $\mathcal{F}$ *contains a forest, then* $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ *for some integer* k.

A General Theorem for Trees

Corollary

If $T \neq K_2$ is a tree with ℓ leaves, then any family $\mathcal{F}$ with $\mathrm{ex}(n, T, \mathcal{F}) = O(n^{\ell})$ has $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^{k})$ for some integer k.

In particular, if $\mathrm{ex}(n, T, \mathcal{F}) = o(n^{\ell})$ then $\mathrm{ex}(n, T, \mathcal{F}) = O(n^{\ell-1})$.

A General Theorem for Trees

Corollary

If $T \neq K_2$ is a tree with ℓ leaves, then any family $\mathcal{F}$ with $\mathrm{ex}(n, T, \mathcal{F}) = O(n^{\ell})$ has $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k.

In particular, if $\mathrm{ex}(n, T, \mathcal{F}) = o(n^{\ell})$ then $\mathrm{ex}(n, T, \mathcal{F}) = O(n^{\ell-1})$.

Proof.

If $\mathcal{F}$ contains a forest then we are done by the previous result.

A General Theorem for Trees

Corollary

If $T \neq K_2$ is a tree with ℓ leaves, then any family $\mathcal{F}$ with $\mathrm{ex}(n, T, \mathcal{F}) = O(n^\ell)$ has $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k.

In particular, if $\mathrm{ex}(n, T, \mathcal{F}) = o(n^\ell)$ then $\mathrm{ex}(n, T, \mathcal{F}) = O(n^{\ell-1})$.

Proof.

If $\mathcal{F}$ contains a forest then we are done by the previous result. Otherwise, the graph G obtained by duplicating each leaf of T a total of n/ℓ times is $\mathcal{F}$-free

A General Theorem for Trees

Corollary

If $T \neq K_2$ is a tree with ℓ leaves, then any family $\mathcal{F}$ with $\mathrm{ex}(n, T, \mathcal{F}) = O(n^\ell)$ has $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k.

In particular, if $\mathrm{ex}(n, T, \mathcal{F}) = o(n^\ell)$ then $\mathrm{ex}(n, T, \mathcal{F}) = O(n^{\ell-1})$.

Proof.

If $\mathcal{F}$ contains a forest then we are done by the previous result. Otherwise, the graph G obtained by duplicating each leaf of T a total of n/ℓ times is $\mathcal{F}$-free and contains $\Omega(n^\ell)$ copies of T. $\square$

A General Theorem for Trees

Theorem (English-S. 2026++)

If T is a tree of diameter d, then for every family of graphs $\mathcal{F}$ with $\mathrm{ex}(n, T, \mathcal{F}) = o(n^{k+1})$, *we have*

$$\mathrm{ex}(n, T, \mathcal{F}) = O(n^{k+\frac{k^2}{d}}).$$

A General Theorem for Trees

Theorem (English-S. 2026++)

If T is a tree of diameter d, then for every family of graphs $\mathcal{F}$ with $\mathrm{ex}(n, T, \mathcal{F}) = o(n^{k+1})$, *we have*

$$\mathrm{ex}(n, T, \mathcal{F}) = O(n^{k+\frac{k^2}{d}}).$$

Theorem (English-S. 2026++)

For every $0 < \epsilon \leq 1$, *if $\mathcal{F}$ is a family of graphs such that*

$$\mathrm{ex}(n, T, \mathcal{F}) = O(n^{(\epsilon v(T))^{1/3}}),$$

then there exists some integer k such that

$$n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+\epsilon}.$$

A General Theorem for Trees

Corollary

For every non-integral number $r \in \mathbb{R} \setminus \mathbb{Z}$, there exists at most finitely many trees T with the property that there exists a family of graphs $\mathcal{F}$ satisfying $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^r)$.

A General Theorem for Trees

Theorem (English-S. 2025+)

Either $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1})$ *or* $\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$.

A General Theorem for Trees

Theorem (English-S. 2025+)

Either $\mathrm{ex}(n, T, \mathcal{F}) = \Omega(n^{k+1})$ *or* $\mathrm{ex}(n, T, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^k)$.

Proposition

This bound is tight for the path P_t *whenever* $t \equiv 2 \mod k-1$.

A General Theorem for Trees

Theorem (English-S. 2026++)

If $t \not\equiv 2 \mod k-1$, *then for every family* $\mathcal{F}$ *either* $\mathrm{ex}(n, P_t, \mathcal{F}) = \Omega(n^{k+1})$ *or*

$$\mathrm{ex}(n, P_t, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^{k-1} \cdot n).$$

A General Theorem for Trees

Theorem (English-S. 2026++)

If $t \not\equiv 2 \mod k-1$, *then for every family* $\mathcal{F}$ *either* $\mathrm{ex}(n, P_t, \mathcal{F}) = \Omega(n^{k+1})$ *or*

$$\mathrm{ex}(n, P_t, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^{k-1} \cdot n).$$

Proposition

This bound is tight for $t \equiv 1 \mod k-1$.

A General Theorem for Trees

Theorem (English-S. 2026++)

If $t \not\equiv 2 \mod k-1$*, then for every family* $\mathcal{F}$ *either* $\mathrm{ex}(n, P_t, \mathcal{F}) = \Omega(n^{k+1})$ *or*

$$\mathrm{ex}(n, P_t, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^{k-1} \cdot n).$$

Proposition

This bound is tight for $t \equiv 1 \mod k-1$.

Our main results rely on two ingredients: a **"stability theorem"** for generalized Turán problems, and variants of the **Helly property** for subtrees of trees.

Stability for Generalized Turán Problems

Stability for Generalized Turán Problems

Proposition (English-Halfpap-Krueger 2024)

For stars $K_{1,t}$, every family of graphs $\mathcal{F}$ either satisfies $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$ *or* $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = O(n)$.

Stability for Generalized Turán Problems

Proposition (English-Halfpap-Krueger 2024)

For stars $K_{1,t}$, every family of graphs $\mathcal{F}$ either satisfies $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$ *or* $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = O(n)$.

Proof.

If $G = K_{1,n-1}$ is $\mathcal{F}$-free, then $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$.

Stability for Generalized Turán Problems

Proposition (English-Halfpap-Krueger 2024)

For stars $K_{1,t}$, every family of graphs $\mathcal{F}$ either satisfies $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$ or $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = O(n)$.

Proof.

If $G = K_{1,n-1}$ is $\mathcal{F}$-free, then $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$. Otherwise, $\mathcal{F}$ must contain some subgraph of some star $K_{1,r}$

Stability for Generalized Turán Problems

Proposition (English-Halfpap-Krueger 2024)

For stars $K_{1,t}$, every family of graphs $\mathcal{F}$ either satisfies $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$ or $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = O(n)$.

Proof.

If $G = K_{1,n-1}$ is $\mathcal{F}$-free, then $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$. Otherwise, $\mathcal{F}$ must contain some subgraph of some star $K_{1,r}$, so $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = O_r(n)$. $\square$

Stability for Generalized Turán Problems

Proposition (English-Halfpap-Krueger 2024)

For stars $K_{1,t}$, every family of graphs $\mathcal{F}$ either satisfies $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$ *or* $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = O(n)$.

Proof.

If $G = K_{1,n-1}$ is $\mathcal{F}$-free, then $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = \Omega(n^t)$. Otherwise, $\mathcal{F}$ must contain some subgraph of some star $K_{1,r}$, so $\mathrm{ex}(n, K_{1,t}, \mathcal{F}) = O_r(n)$. □

We want to generalize this idea by saying that for *any* graph H, there exists some "simple" family $\mathcal{F}_H$ such that the behavior of $\mathrm{ex}(n, H, \mathcal{F})$ depends on how $\mathcal{F}$ "interacts" with $\mathcal{F}_H$.

Stability for Generalized Turán Problems

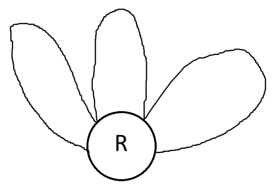
Definition

Given a graph H, a subset $R \subseteq V(H)$, and an integer q, we define the sunflower-power H_R^q to be the graph obtained by taking q copies of H which all agree on R and which are otherwise disjoint.

Stability for Generalized Turán Problems

Definition

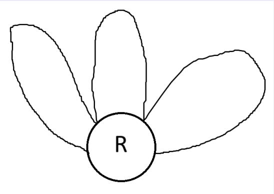
Given a graph H, a subset $R \subseteq V(H)$, and an integer q, we define the sunflower-power H_R^q to be the graph obtained by taking q copies of H which all agree on R and which are otherwise disjoint.



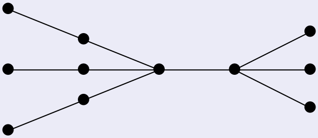
Stability for Generalized Turán Problems

Definition

Given a graph H, a subset $R \subseteq V(H)$, and an integer q, we define the sunflower-power H_R^q to be the graph obtained by taking q copies of H which all agree on R and which are otherwise disjoint.



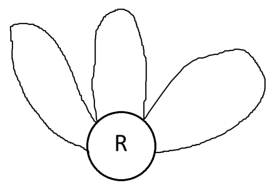
$(P_5)^3_{\{x_3,x_4\}}$



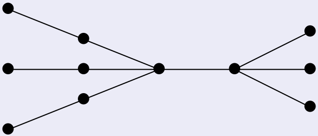
Stability for Generalized Turán Problems

Definition

Given a graph H, a subset $R \subseteq V(H)$, and an integer q, we define the sunflower-power H_R^q to be the graph obtained by taking q copies of H which all agree on R and which are otherwise disjoint.



$(P_5)^3_{\{x_3,x_4\}}$



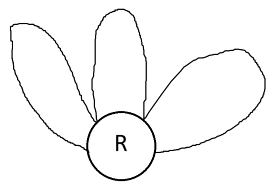
We define

$$\mathcal{F}^q_{H,k} = \{H_R^q : H - R \text{ has at least } k \text{ connected components}\}.$$

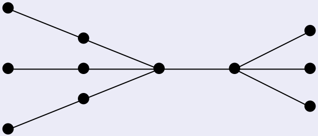
Stability for Generalized Turán Problems

Definition

Given a graph H, a subset $R \subseteq V(H)$, and an integer q, we define the sunflower-power H_R^q to be the graph obtained by taking q copies of H which all agree on R and which are otherwise disjoint.



$(P_5)^3_{\{x_3,x_4\}}$



We define

$$\mathcal{F}^q_{H,k} = \{H^q_R : H - R \text{ has at least } k \text{ connected components}\}.$$

Claim

Every graph in $\mathcal{F}^q_{H,k}$ has at least q^k copies of H.

Stability for Generalized Turán Problems

Proposition (Key Observation)

For every $H, k, \mathcal{F}$, either $\mathrm{ex}(n, H, \mathcal{F}) = \Omega(n^{k+1})$ or there exists some q such that every $\mathcal{F}$-free graph is $\mathcal{F}^{q}_{H,k+1}$-free.

Stability for Generalized Turán Problems

Proposition (Key Observation)

For every $H, k, \mathcal{F}$, either $\mathrm{ex}(n, H, \mathcal{F}) = \Omega(n^{k+1})$ or there exists some q such that every $\mathcal{F}$-free graph is $\mathcal{F}^q_{H,k+1}$-free.

Corollary

If $\mathrm{ex}(n, H, \mathcal{F}^q_{H,k+1}) = O(n^\beta)$, then every family $\mathcal{F}$ either satisfies $\mathrm{ex}(n, H, \mathcal{F}) = \Omega(n^{k+1})$ or $\mathrm{ex}(n, H, \mathcal{F}) = O(n^\beta)$.

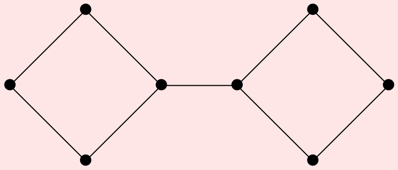
Proof Sketch of Main Result

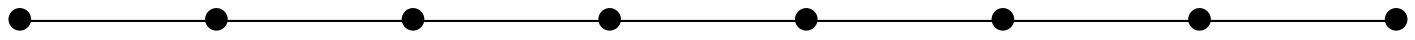
Proof Sketch of Main Result

To prove our main theorem, we in particular need to solve the following:

Problem

Show that $\mathrm{ex}(n, P_8, F) = O(n^{9/2})$ where F consists of two C_4's connected by an edge.



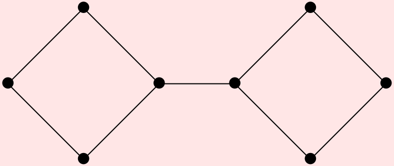


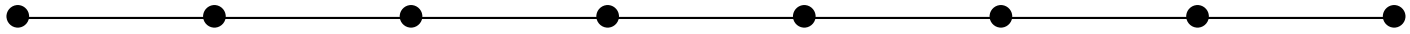
Proof Sketch of Main Result

To prove our main theorem, we in particular need to solve the following:

Problem

Show that $\mathrm{ex}(n, P_8, F) = O(n^{9/2})$ where F consists of two C_4's connected by an edge.





Fact

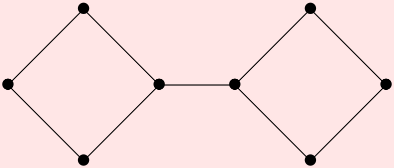
This is easy to do if we replace F with C_4

Proof Sketch of Main Result

To prove our main theorem, we in particular need to solve the following:

Problem

Show that $\mathrm{ex}(n, P_8, F) = O(n^{9/2})$ where F consists of two C_4's connected by an edge.



Fact

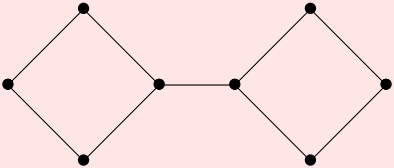
This is easy to do if we replace F with C_4

Proof Sketch of Main Result

To prove our main theorem, we in particular need to solve the following:

Problem

Show that $\mathrm{ex}(n, P_8, F) = O(n^{9/2})$ where F consists of two C_4's connected by an edge.



Fact

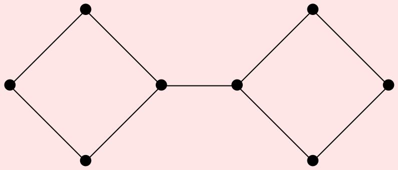
This is easy to do if we replace F with C_4: one gets an upper bound of $e(G)^3$

Proof Sketch of Main Result

To prove our main theorem, we in particular need to solve the following:

Problem

Show that $\mathrm{ex}(n, P_8, F) = O(n^{9/2})$ where F consists of two C_4's connected by an edge.



Fact

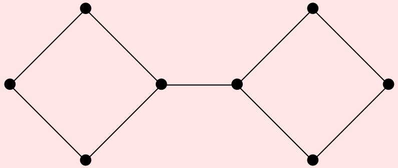
This is easy to do if we replace F with C_4: one gets an upper bound of $e(G)^3 = O((n^{3/2})^3)$

Proof Sketch of Main Result

To prove our main theorem, we in particular need to solve the following:

Problem

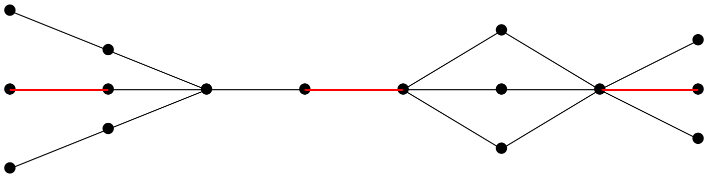
Show that $\mathrm{ex}(n, P_8, F) = O(n^{9/2})$ where F consists of two C_4's connected by an edge.



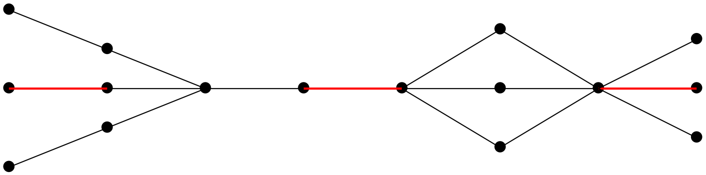
Fact

This is easy to do if we replace F with C_4: one gets an upper bound of $e(G)^3 = O((n^{3/2})^3)$, and any solution to the problem must generalize such an argument.

Proof Sketch of Main Result



Proof Sketch of Main Result



Whatever set of edges E we choose to identify a given copy K in a graph, the edges of E **must** intersect every subtree $K' \subseteq K$ which has "many extensions."

Proof Sketch of Main Result

Question

Given a set of subtrees $\mathcal{T}$ of P_8, when can we guarantee that there exist a set of 3 edges E which intersect all of the vertex sets of these subtrees?

Proof Sketch of Main Result

Question

Given a set of subtrees $\mathcal{T}$ of P_8, when can we guarantee that there exist a set of 3 edges E which intersect all of the vertex sets of these subtrees?

Theorem (Helly Property for Trees)

If $\mathcal{T}$ is a set of subtrees of a tree T such that the vertex sets of the subtrees pairwise intersect, then there exists a vertex $v \in V(T)$ which intersects the vertex set of every subtree of $\mathcal{T}$.

Proof Sketch of Main Result

Theorem (Helly Property for Trees Restated)

If $\mathcal{T}$ is a set of subtrees of a tree T such that for every subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most 2 there exists a vertex v' which intersects the vertex set of each $T' \in \mathcal{T}'$, then there exists a vertex v which intersects the vertex set of each $T' \in \mathcal{T}$.

Proof Sketch of Main Result

Theorem (Helly Property for Trees Restated)

If $\mathcal{T}$ is a set of subtrees of a tree T such that for every subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most 2 there exists a vertex v' which intersects the vertex set of each $T' \in \mathcal{T}'$, then there exists a vertex v which intersects the vertex set of each $T' \in \mathcal{T}$.

Theorem (English-S. 2025+)

If $\mathcal{T}$ is a set of subtrees of a tree T such that for every subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most $k+1$ there exists a set of at most k edges E' which intersects the vertex set of each $T' \in \mathcal{T}'$, then there exists a set of at most k edges E which intersects the vertex set of each $T' \in \mathcal{T}$.

Proof Sketch of Main Result

Theorem (Helly Property for Trees Restated)

If $\mathcal{T}$ is a set of subtrees of a tree T such that for every subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most 2 there exists a vertex v' which intersects the vertex set of each $T' \in \mathcal{T}'$, then there exists a vertex v which intersects the vertex set of each $T' \in \mathcal{T}$.

Theorem (English-S. 2025+)

If $\mathcal{T}$ is a set of subtrees of a tree T such that for every subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most $k+1$ there exists a set of at most k edges E' which intersects the vertex set of each $T' \in \mathcal{T}'$, then there exists a set of at most k edges E which intersects the vertex set of each $T' \in \mathcal{T}$.

This result implies the vertex-Helly result (and also König's Theorem for trees).

Proof Sketch of Main Result

Theorem (English-S. 2025+)

If $\mathcal{T}$ is a set of subtrees of a tree T such that for every subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most $k+1$ there exists a set of at most k edges E' which intersects the vertex set of each $T' \in \mathcal{T}'$, then there exists a set of at most k edges E which intersects the vertex set of each $T' \in \mathcal{T}$.

Proof Sketch of Main Result

Theorem (English-S. 2025+)

If $\mathcal{T}$ is a set of subtrees of a tree T such that for every subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most $k+1$ there exists a set of at most k edges E' which intersects the vertex set of each $T' \in \mathcal{T}'$, then there exists a set of at most k edges E which intersects the vertex set of each $T' \in \mathcal{T}$.

Claim

The "highly extendable" subtrees $\mathcal{T}_K$ for a copy K satisfy the conditions of the edge-Helly theorem whenever G is $\mathcal{F}^q_{T,k+1}$-free.

Proof Sketch of Main Result

Theorem (English-S. 2026++)

If $t \not\equiv 2 \mod k-1$, *then for every family* $\mathcal{F}$ *either* $\mathrm{ex}(n, P_t, \mathcal{F}) = \Omega(n^{k+1})$ *or*

$$\mathrm{ex}(n, P_t, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^{k-1} \cdot n).$$

Proof Sketch of Main Result

Theorem (English-S. 2026++)

If $t \not\equiv 2 \mod k-1$, then for every family $\mathcal{F}$ either $\mathrm{ex}(n, P_t, \mathcal{F}) = \Omega(n^{k+1})$ *or*

$$\mathrm{ex}(n, P_t, \mathcal{F}) = O(\mathrm{ex}(n, \mathcal{F})^{k-1} \cdot n).$$

Theorem (English-S. 2026++)

If $\mathcal{T}$ is a collection of subtrees such that for any subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most $2k$ there exists a set E' of at most $k-1$ edges and a vertex v' which intersects the vertex set of each $T' \in \mathcal{T}'$, then one can find E, v which intersect the vertex set of every $T' \in \mathcal{T}$.

Proof Sketch of Main Result

Theorem (English-S. 2026++)

If $t \not\equiv 2 \mod k-1$, then for every family $\mathcal{F}$ either $\text{ex}(n, P_t, \mathcal{F}) = \Omega(n^{k+1})$ *or*

$$\text{ex}(n, P_t, \mathcal{F}) = O(\text{ex}(n, \mathcal{F})^{k-1} \cdot n).$$

Theorem (English-S. 2026++)

If $\mathcal{T}$ is a collection of subtrees such that for any subcollection $\mathcal{T}' \subseteq \mathcal{T}$ of size at most $2k$ there exists a set E' of at most $k-1$ edges and a vertex v' which intersects the vertex set of each $T' \in \mathcal{T}'$, then one can find E, v which intersect the vertex set of every $T' \in \mathcal{T}$.

It is **not** true that the set of highly-extendable subtrees $\mathcal{T}_K$ always satisfy these Helly conditions.

Further Refinements

Our refinement for paths P_t is tight for $t \equiv 1 \mod k-1$.

Further Refinements

Our refinement for paths P_t is tight for $t \equiv 1 \mod k-1$.

Conjecture

If $t \geq 2k+1$ and $t-3 \equiv r \mod k-1$, then any $\mathcal{F}^q_{P_t,k+1}$-free graph has at most

$$O(e(G)^{r+2} \cdot n^{k-2-r})$$

copies of P_t.

Further Refinements

Our refinement for paths P_t is tight for $t \equiv 1 \mod k-1$.

Conjecture

If $t \geq 2k+1$ and $t-3 \equiv r \mod k-1$, then any $\mathcal{F}^q_{P_t,k+1}$-free graph has at most

$$O(e(G)^{r+2} \cdot n^{k-2-r})$$

copies of P_t.

Problem

What (optimal) conditions guarantee that a collection of subtrees $\mathcal{T}$ can be pierced by $r+2$ edges and $k-2-r$ vertices?

Realizable Exponents

We've seen a lot of ways that $\mathrm{ex}(n, H, \mathcal{F})$ *can't* behave like; what about the ways it *can* behave?

Realizable Exponents

We've seen a lot of ways that $\mathrm{ex}(n, H, \mathcal{F})$ *can't* behave like; what about the ways it *can* behave?

Definition

We say that a rational number r is *realizable* for a graph H if there exists a (finite) family of graphs $\mathcal{F}$ such that $\mathrm{ex}(n, H, \mathcal{F}) = \Theta(n^r)$.

Realizable Exponents

We've seen a lot of ways that $\mathrm{ex}(n, H, \mathcal{F})$ *can't* behave like; what about the ways it *can* behave?

Definition

We say that a rational number r is *realizable* for a graph H if there exists a (finite) family of graphs $\mathcal{F}$ such that $\mathrm{ex}(n, H, \mathcal{F}) = \Theta(n^r)$.

Theorem (Bukh-Conlon 2018)

Every rational in $[1, 2]$ is realizable for $H = K_2$.

Realizable Exponents

Theorem (English-Halfpap-Krueger 2024)

Every rational in $[1, t]$ is realizable for $H = K_t$.

Realizable Exponents

Theorem (English-Halfpap-Krueger 2024)

Every rational in $[1, t]$ *is realizable for* $H = K_t$.

Theorem (English-Halfpap-Krueger 2024)

Every rational in $[t, t+1]$ *is realizable for* $H = K_{1,t}$ *and no rational in* $(1, t)$ *is.*

Realizable Exponents

Theorem (English-S. 2025+)

For every graph H of maximum degree Δ, every rational in the range

$$\left[v(H) - \frac{e(H)}{2\Delta^2}, v(H)\right]$$

is realizable.

Realizable Exponents

Theorem (English-S. 2025+)

For every graph H of maximum degree Δ, every rational in the range

$$\left[v(H) - \frac{e(H)}{2\Delta^2}, v(H)\right]$$

is realizable.

Theorem (English-S. 2025+)

For every tree T of maximum degree Δ, every rational in the range

$$\left[v(T) - \frac{e(T)}{\Delta}, v(T)\right]$$

is realizable.

This is best possible for stars.

Open Problems

Question

Is it the case that every rational in $[\alpha(H), v(H)]$ is realizable?

This is true and best possible for cliques and stars.

Open Problems

Question

Is it the case that every rational in $[\alpha(H), v(H)]$ is realizable?

This is true and best possible for cliques and stars.

Conjecture

For every graph H, every rational in

$$\left[v(H) - \frac{e(H)}{\Delta}, v(H)\right]$$

is realizable.

Open Problems

Say that a rational r is strongly realizable with respect to H if there exists a single graph F such that $\mathrm{ex}(n, H, F) = \Theta(n^r)$.

Open Problems

Say that a rational r is strongly realizable with respect to H if there exists a single graph F such that $\mathrm{ex}(n, H, F) = \Theta(n^r)$.

Question

Could the following be true for every graph H?

(a) Every rational in $[\alpha(H), v(H)]$ is strongly realizable.

Open Problems

Say that a rational r is strongly realizable with respect to H if there exists a single graph F such that $\mathrm{ex}(n, H, F) = \Theta(n^r)$.

Question

Could the following be true for every graph H?

(a) Every rational in $[\alpha(H), v(H)]$ is strongly realizable.

(b) Below $\alpha(H)$ there exist only finitely many strongly realizable exponents.

Open Problems

Problem

Show that for any H that there exist infinitely many strongly realizable rationals exponents.

Open Problems

Problem

Show that for any H that there exist infinitely many strongly realizable rationals exponents.

Problem

Show that there exists a rational r and integer t_0 such that for all $t \geq t_0$, we have

$$\mathrm{ex}(n, K_{3,3}, K_{2,t}) = \Theta(n^r).$$

This is needed to show that only $O(1)$ rationals below $\alpha(K_{3,3})$ are strongly realizable.

Open Problems

Problem

Is it the case that for all trees $T \neq K_2$, there exists an $\epsilon > 0$ such that if $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\epsilon})$ then $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k?

Recall that our main theorem implies $n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+k\epsilon}$; note also that this result is true for stars.

Open Problems

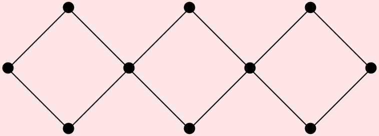
Problem

Is it the case that for all trees $T \neq K_2$, there exists an $\epsilon > 0$ such that if $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\epsilon})$ then $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k?

Recall that our main theorem implies $n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+k\epsilon}$; note also that this result is true for stars.

Problem

Give an (easy) proof that $\mathrm{ex}(n, P_9, F) = O(n^5)$ where F is the following.



This bound is of the form $\mathrm{ex}(n, F)^2 n^2$.

Open Problems

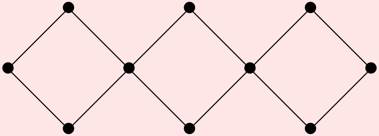
Problem

Is it the case that for all trees $T \neq K_2$, there exists an $\epsilon > 0$ such that if $\mathrm{ex}(n, \mathcal{F}) = O(n^{1+\epsilon})$ then $\mathrm{ex}(n, T, \mathcal{F}) = \Theta(n^k)$ for some integer k?

Recall that our main theorem implies $n^k \ll \mathrm{ex}(n, T, \mathcal{F}) \ll n^{k+k\epsilon}$; note also that this result is true for stars.

Problem

Give an (easy) proof that $\mathrm{ex}(n, P_9, F) = O(n^5)$ where F is the following.



This bound is of the form $\mathrm{ex}(n, F)^2 n^2$. I'm willing to offer \$1 for an answer to any of these last three problems.

Helly Theorems

Helly Theorems

Definition

- We say that a set of graphs $\mathcal{H}$ pierces a collection of subtrees $\mathcal{T}$ of a tree T if there exists a subgraph $S \subseteq T$ isomorphic to a graph in $\mathcal{H}$ such that $V(S) \cap V(T') \neq \emptyset$ for all $T' \in \mathcal{T}$.

Helly Theorems

Definition

- We say that a set of graphs $\mathcal{H}$ pierces a collection of subtrees $\mathcal{T}$ of a tree T if there exists a subgraph $S \subseteq T$ isomorphic to a graph in $\mathcal{H}$ such that $V(S) \cap V(T') \neq \emptyset$ for all $T' \in \mathcal{T}$.
- We say that $\mathcal{H}$ is m-Helly if having $\mathcal{H}$ pierce any $\mathcal{T}' \subseteq \mathcal{T}$ of size at most m implies that $\mathcal{H}$ pierces $\mathcal{T}$.

Helly Theorems

Definition

- We say that a set of graphs $\mathcal{H}$ pierces a collection of subtrees $\mathcal{T}$ of a tree T if there exists a subgraph $S \subseteq T$ isomorphic to a graph in $\mathcal{H}$ such that $V(S) \cap V(T') \neq \emptyset$ for all $T' \in \mathcal{T}$.
- We say that $\mathcal{H}$ is m-Helly if having $\mathcal{H}$ pierce any $\mathcal{T}' \subseteq \mathcal{T}$ of size at most m implies that $\mathcal{H}$ pierces $\mathcal{T}$.
- We let $\mu(\mathcal{H})$ be the smallest number m such that $\mathcal{H}$ is m-Helly (with $\mu(\mathcal{H}) = \infty$ if no such number exists).

Helly Theorems

Definition

- We say that a set of graphs $\mathcal{H}$ pierces a collection of subtrees $\mathcal{T}$ of a tree T if there exists a subgraph $S \subseteq T$ isomorphic to a graph in $\mathcal{H}$ such that $V(S) \cap V(T') \neq \emptyset$ for all $T' \in \mathcal{T}$.
- We say that $\mathcal{H}$ is m-Helly if having $\mathcal{H}$ pierce any $\mathcal{T}' \subseteq \mathcal{T}$ of size at most m implies that $\mathcal{H}$ pierces $\mathcal{T}$.
- We let $\mu(\mathcal{H})$ be the smallest number m such that $\mathcal{H}$ is m-Helly (with $\mu(\mathcal{H}) = \infty$ if no such number exists).

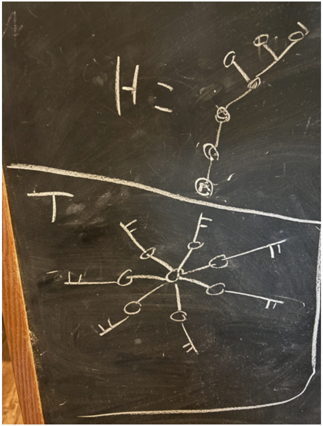
Problem

Determine or bound $\mu(\mathcal{H})$ for families of graphs $\mathcal{H}$.

Helly Theorems

Fact

$\mu(H) = \infty$ for almost all trees H for silly reasons.



Helly Theorems

Theorem (English-S. 2026++)

If H is a tree with ℓ leaves, then

$$\Delta + 1 \leq \mu(\{H, K_1\}) \leq \ell + 1.$$

Helly Theorems

Theorem (English-S. 2026++)

If H is a tree with ℓ leaves, then

$$\Delta + 1 \leq \mu(\{H, K_1\}) \leq \ell + 1.$$

Theorem (English-S. 2026++)

If $\mathcal{H}$ is any finite family of forests which is closed under replacing connected components with K_1, then

$$\mu(\mathcal{H}) < \infty.$$

Helly Theorems

Theorem (English-S. 2026++)

If H is a tree with ℓ leaves, then

$$\Delta + 1 \leq \mu(\{H, K_1\}) \leq \ell + 1.$$

Theorem (English-S. 2026++)

If $\mathcal{H}$ is any finite family of forests which is closed under replacing connected components with K_1, then

$$\mu(\mathcal{H}) < \infty.$$

Problem

Determine $\mu(\mathcal{H}_{a,b})$ where $\mathcal{H}_{a,b}$ is the set of all graphs which can be written as the union of a edges and b vertices.