Ballot Permutations and Odd Order Permutations

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5/17/19

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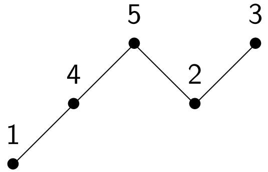

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Given a cycle $\bar{c} = (c_1, \ldots, c_k)$, we let $\mathrm{asc}'(\bar{c})$ denote the number of cyclic ascents of $\bar{c}$. That is, the number of ascents in $c_1 c_2 \cdots c_k c_1$. Similarly define $\mathrm{des}'(\bar{c})$.

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Let $M(\bar{c}) = \min\{\mathrm{asc}'(\bar{c}), \mathrm{des}'(\bar{c})\}$. For example, $M(48256) = 2$. Given a permutation $\pi = \bar{c}_1 \cdots \bar{c}_k$, we define $M(\pi) = \sum_{i=1}^{k} M(\bar{c}_i)$. For example, if $\pi = (139)(48256)(7)$, then

$$M(\pi) = 1 + 2 + 0 = 3.$$

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$$\begin{aligned} p(3,0) &= b(3,0) = 1, \\ p(3,1) &= b(3,1) = 2, \\ p(4,0) &= b(4,0) = 1, \\ p(4,1) &= b(4,1) = 8. \end{aligned}$$

The M Statistic

b(n,d)	d=0	d=1	d=2	d=3	d=4
n=1	1	0	0	0	0
n=2	1	0	0	0	0
n=3	1	2	0	0	0
n=4	1	8	0	0	0
n=5	1	22	22	0	0
n=6	1	52	172	0	0
n=7	1	114	856	604	0
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Conjecture

For all n, d, *we have* $p(n, d) = b(n, d)$.

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What are $P(n,1)$ and $B(n,1)$ again?

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Idea for the bijection: use the non-trivial odd cycle of $\pi \in P(n, 1)$ as the "backbone" for the permutation $\phi(\pi)$. In particular, use the largest element of this cycle as the unique descent of $\phi(\pi)$.

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Observe that x and y consist of only ascents (because the c_i are increasing), and that x and y are non-empty because $k \geq 1$ (so the word starts with an ascent and has a descent at position $k+1$).

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So far we have a word with exactly one descent and which begins with an ascent. We want to insert the remaining elements of $[n]$ into this word without disrupting this structure.
Say we've currently constructed the word w and that the largest letter we haven't inserted yet is k. How can we insert k into w without creating an extra descent? If $k = n$, put it at the end of the word, otherwise put k right before $k + 1$. For example, with σ we have

$$25846 \rightarrow 258469 \rightarrow 2578469 \rightarrow 25783469$$

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This defines $\phi(\pi)$ if $\pi = (c_1 \cdots c_{2k+1})$ and $c_i < c_{i+1}$ for all i, but what if we were given $\tilde{\pi} = (c_{2k+1} \cdots c_1)$?

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One can verify that this is always an element of $B(n, 1)$, and this defines our map $\phi : P(n, 1) \to B(n, 1)$.

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$$125783469 \rightarrow (24568).$$

It turns out that when $\pi \in B(n, 1)$ then π (or its flipped version) always has an odd number of consecutive runs, so this returns something in $P(n)$, and one can check that it is in fact in $P(n, 1)$. □

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Theorem (S. (2018))

For all $n \geq 4$,

$$p(n,1) = b(n,1) = 2E(n-1,1),$$

$$p(n,2) = b(n,2) = 3E(n-1,2) - 2\binom{n}{3} + \binom{n}{2} - 1,$$

$$p(n,3) = b(n,3) =$$

$$4E(n-1,3) - \left(\binom{n}{3} - \binom{n}{2} + 4\right)2^{n-2} - 22\binom{n}{5} + 16\binom{n}{4} - 4\binom{n}{3} + 2n.$$

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To get formulas for ballot permutations, one can start with permutations that begin with an ascent, and then use ideas from Shevelev to get rid of the permutations that aren't ballot.

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Theorem (Bidkhori, Sullivant (2011), S. (2018))

For all $n \geq 0$, we have $p(2n+1, n) = b(2n+1, n) = EC(n)$. Moreover, there exists an explicit bijection from $P(2n+1, n)$ to $B(2n+1, n)$.

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Theorem (S. (2018))

For all $n \geq 1$, we have $p(2n, n-1) = b(2n, n-1) =$

$$\frac{1}{2} \sum_{k \geq 1,\ k\ odd} \binom{2n}{k} EC\left(\frac{k-1}{2}\right) EC\left(\frac{2n-k-1}{2}\right).$$

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Solving the $d = 2$ case may show how to deal with multiple non-trivial odd cycles in general, which could give insight into a general bijection.

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Problem

Determine the generating function

$$C(x, y) = \sum_{n} \sum_{d \leq n-1} E(2n, d) \frac{x^{2n}}{(2n)!} y^d.$$