Polynomial Relations of Matrices of Graphs

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Spectral Graph Theory

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Main Idea

G

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$$G \to M_G \ (\text{or } M)$$

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$$G \to M_G \text{ (or } M) \to \{\lambda_1, \ldots, \lambda_n\}$$

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Main Idea

$$G \to M_G \text{ (or } M) \to \{\lambda_1, \ldots, \lambda_n\} \to \text{Properties of } G$$

The Adjacency Matrix A

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Definition

Define the adjacency matrix A by

$$A_{ij} = \begin{cases} 1 & ij \in E(G) \\ 0 & ij \notin E(G) \end{cases}$$

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Theorem

If $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of A, then the total number of closed walks (walks from a vertex to itself) of length k in G is

$$\lambda_1^k + \cdots + \lambda_n^k.$$

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$$(A^k)_{11} + \cdots + (A^k)_{nn} = \operatorname{Tr} A^k = \lambda_1^k + \cdots + \lambda_n^k.$$

□

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$$L = D - A.$$

Example

Let P_3 denote the path on 3 vertices. Then

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

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Theorem

Let $\{\mu_1, \ldots, \mu_n\}$ be the eigenvalues of L with $\mu_n = 0$. If $t(G)$ denotes the number of spanning trees of G then

$$t(G) = \frac{1}{n}\mu_1\mu_2\cdots\mu_{n-1}.$$

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If G is d-regular then $D = dI$ and we have $L = dI - A$, or equivalently

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Proposition

If G is d-regular and $\{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of A then $\{d - \lambda_1, \ldots, d - \lambda_n\}$ is the set of eigenvalues of L. Conversely, if $\{\mu_1, \ldots, \mu_n\}$ is the set of eigenvalues of L then $\{d - \mu_1, \ldots, d - \mu_N\}$ is the set of eigenvalues of A.

Regular Graphs

Main Idea

Given the relation $A = dI - L$ we can translate from eigenvalues of A to eigenvalues of L and vice versa.

Biregular Graphs

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Definition

A graph G is said to be (d_1, d_2)-biregular if it is bipartite with vertex partition $V = V_1 \sqcup V_2$ such that for all $v \in V_1$ we have $d(v) = d_1$ and for all $w \in V_2$ we have $d(w) = d_2$.

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Example

$K_{n,m}$ is biregular: it has an obvious bipartition, all the vertices in one set have degree m and all the other vertices have degree n.

Biregular Graphs

Remark

If G is (d_1, d_2)-biregular then we claim that

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Theorem (S.)

Given the relation $A^2 = (L - d_1 I)(L - d_2 I)$ we can translate from eigenvalues of A to eigenvalues of L and vice versa.

$A^r = f(L)$

Question

Are there "triregular" graphs G such that

$$A^3 = f(L),$$

where f is some polynomial?

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Theorem (S.)

No.

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Theorem (S.)

Let G be a connected graph. If there exists a positive integer r and polynomial f such that

$$A^r = f(L),$$

then G is regular or biregular.

$A^r = f(L)$

Theorem (Perron-Frobenius)

There exists an eigenvector $\tilde{v}$ of A such that

1. *Every coordinate of $\tilde{v}$ is non-zero.*
2. *The corresponding eigenvalue Λ has multiplicity 1 in A.*

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Lemma

If $\tilde{v}$ is also an eigenvector of L, then G is regular.

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If $\tilde{v}$ is also an eigenvector of L, then G is regular.

Proof.

If $L\tilde{v} = \mu\tilde{v}$ then $D\tilde{v} = A\tilde{v} + L\tilde{v} = (\Lambda + \mu)\tilde{v}$.

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If $L\tilde{v} = \mu\tilde{v}$ then $D\tilde{v} = A\tilde{v} + L\tilde{v} = (\Lambda + \mu)\tilde{v}$. Thus $D_{ii} = \Lambda + \mu$ whenever $\tilde{v}_i \neq 0$.

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If $\tilde{v}$ is also an eigenvector of L, then G is regular.

Proof.

If $L\tilde{v} = \mu\tilde{v}$ then $D\tilde{v} = A\tilde{v} + L\tilde{v} = (\Lambda + \mu)\tilde{v}$. Thus $D_{ii} = \Lambda + \mu$ whenever $\tilde{v}_i \neq 0$. But every coordinate is non-zero, so we must have $D = (\Lambda + \mu)I$, so G is $(\Lambda + \mu)$-regular. □

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Case 1: r odd.

If $A^r = f(L)$ with r odd, then the multiplicity of Λ^r of A^r is 1 (if r is even and $-\Lambda$ is an eigenvalue of A then this would not be true).

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An orthonormal basis of eigenvectors for L is an orthonormal basis of eigenvectors for A^r, so there must be an eigenvector v of L such that $A^r v = \Lambda^r v$.

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An orthonormal basis of eigenvectors for L is an orthonormal basis of eigenvectors for A^r, so there must be an eigenvector v of L such that $A^r v = \Lambda^r v$. But Λ^r has multiplicity 1, so v must be a scaler multiple of $\tilde{v}$.

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An orthonormal basis of eigenvectors for L is an orthonormal basis of eigenvectors for A^r, so there must be an eigenvector v of L such that $A^r v = \Lambda^r v$. But Λ^r has multiplicity 1, so v must be a scaler multiple of $\tilde{v}$. By our lemma, G must be regular. $\square$

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If $A^{2k} = (A^2)^k = f(L)$, then an orthonormal basis of eigenvectors for L will be a basis of eigenvectors for A^2 (since its spectrum is non-negative).

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If $A^{2k} = (A^2)^k = f(L)$, then an orthonormal basis of eigenvectors for L will be a basis of eigenvectors for A^2 (since its spectrum is non-negative). In particular, $\mathbf{1}$ is an eigenvector for L so it will also be an eigenvector for A^2. □

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Lemma

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Proof.

Claim: $(A^2\mathbf{1})_i = \sum_{j:ij\in E(G)} d_j$.

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Claim: $(A^2\mathbf{1})_i = \sum_{j:ij \in E(G)} d_j$. Thus $\mathbf{1}$ being an eigenvector for A^2 is equivalent to the statement that there exists a λ such that $\sum_{j:ij \in E(G)} d_j = \lambda$ for all i. Assume this is the case.

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We conclude that the inequalities are equalities. In particular, every vertex of minimum degree is adjacent only to vertices of maximum degree and vice versa. If $d \neq D$, this means G is biregular. $\square$

$X^r = f(Y)$

Question

Let $Q := D + A$ and $\mathcal{L} := D^{-1/2} L D^{-1/2}$ be the signless Laplacian and normalized Laplacian respectively.

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Let $Q := D + A$ and $\mathcal{L} := D^{-1/2} L D^{-1/2}$ be the signless Laplacian and normalized Laplacian respectively.
When do there exist r, f such that $X^r = f(Y)$ for X, Y two of the four matrices $A,\ L,\ Q,\ \mathcal{L}$?

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Table: Graphs Satisfying $X^r = f(Y)$

X/Y	A	L	Q	$\mathcal{L}$
A		Reg, Bireg	Reg, Bireg	???
L	Reg		Reg	Reg
Q	Reg	Reg		Reg
$\mathcal{L}$	Reg, Bireg	Reg	Reg	

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$$f(x) = m_X(x),\ g(x) = m_Y(x),$$

where $m_M(x)$ denotes the minimal polynomial of M.

$f(X) = g(Y)$

Definition

We will say that a relation $f(X) = g(Y)$ is *proper* whenever f and g are polynomials satisfying $1 \leq \deg(f) < \deg(m_X)$ and $1 \leq \deg(g) < \deg(m_Y)$.

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Given two matrices X, Y associated to a graph G, when does there exist a proper relation

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$f(X) = g(Y)$: Families of Graphs

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Question

Given a family of graphs $\mathcal{F}$ and two matrices $X,\ Y$ associated to graphs, does there exist a proper relations $f_G(X_G) = g_G(Y_G)$ for all $G \in \mathcal{F}$, and if so, what do the polynomials look like?

$f(X) = g(Y)$: Families of Graphs

Question

If $G = P_n$, does there exist, for all n, polynomials f_n, g_n such that $f_n(A) = g_n(L)$ for all n? If so, what do these polynomials look like?

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$$n = 2: \; I - A = L$$

$$n = 3: \; I - 1/2A^2 = 3/2L - 1/2L^2$$

$$n = 4: \; I + 2A - A^3 = 6L - 5L^2 + L^3$$

$$n = 5: \; A^2 - 1/3A^4 = 10/3L - 5L^2 + 7/3L^3 - 1/3L^4$$

$$n = 6: \; I - 3A + 4A^3 - A^5 = 15L - 35L^2 + 28L^3 - 9L^4 + L^5$$

$f(X) = g(Y)$: Families of Graphs

Question

Given X, Y what are the graphs such that there exists no proper relation $f(X) = g(Y)$?

$f(X) = g(Y)$: Families of Graphs

Proposition

If K_n' denotes the complete graph on n vertices with one edge removed, then there exists no proper relation $f(Q) = g(L)$ for $n \geq 4$.

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Sketch of Proof.

For any fixed G one can generate the matrices $\{Q^k\}$ and $\{L^k\}$ such that all proper polynomials are linear combinations of these matrices.

$f(X) = g(Y)$: Families of Graphs

Proposition

If K'_n denotes the complete graph on n vertices with one edge removed, then there exists no proper relation $f(Q) = g(L)$ for $n \geq 4$.

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For any fixed G one can generate the matrices $\{Q^k\}$ and $\{L^k\}$ such that all proper polynomials are linear combinations of these matrices. If these matrices are linearly independent, then no proper relation exists.

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Sketch of Proof.

For any fixed G one can generate the matrices $\{Q^k\}$ and $\{L^k\}$ such that all proper polynomials are linear combinations of these matrices. If these matrices are linearly independent, then no proper relation exists.
To prove this for the whole family of K_n' graphs, one observes that the minimal polynomials of Q and L both have degree 3.

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If K'_n denotes the complete graph on n vertices with one edge removed, then there exists no proper relation $f(Q) = g(L)$ for $n \geq 4$.

Sketch of Proof.

For any fixed G one can generate the matrices $\{Q^k\}$ and $\{L^k\}$ such that all proper polynomials are linear combinations of these matrices. If these matrices are linearly independent, then no proper relation exists.

To prove this for the whole family of K'_n graphs, one observes that the minimal polynomials of Q and L both have degree 3. Thus one only has to ask if there exists constants such that $aQ^2 + bQ = cL^2 + dL + eI$, and one can verify that no such constants exist if n is at least 4. □

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Example

If $X = A$, $Y = L$ and we require $f(x) = x^r$ for some r, then the graphs satisfying $f(A) = g(L)$ are the regular and biregular graphs.

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Given X, Y, what graphs satisfy $f(X) = f(Y)$ for f proper?

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Given X, Y, what graphs satisfy $f(X) = -f(Y)$ for f proper?

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Conjecture

If $f(A) = -f(L)$ with $-f$ proper then G is regular and A has at least 4 distinct eigenvalues (i.e. it's not "strongly regular").

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- How are relations affected by graph operations (cones, deleting edges)?

The End

Thank You!