Cycle-free Subgraphs of Random Hypergraphs.

Sam Spiro, UC San Diego.

Joint work with Jacques Verstraëte

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Let $\mathcal{F}$ be a family of graphs. We define the Turán number (or extremal number) of $\mathcal{F}$ to be the maximum number of edges in an $\mathcal{F}$-free graph on n vertices, and we'll denote this quantity by $\text{ex}(n, \mathcal{F})$.

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Conjecture (Erdős-Simonovits, 1983)

$\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2\ell}\}) = \Theta(n^{1+1/\ell})$.

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If every graph of $\mathcal{F}$ is non-bipartite, then this problem has essentially been solved independently by Conlon-Gowers and Schacht. Thus we will focus our attention on the case when $\mathcal{F}$ contains bipartite graphs, and in general this problem is unsolved.

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where m is some function of n, p. In general if the probability of a sequence of events A_n tends to 1 we say that the event happens asymptotically almost surely or simply a.a.s.

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Proof.

We use a first moment method. Define the random variable X to be the number of $\mathcal{F}$-free subgraphs of $G_{n,p}$ on m edges. Then

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Thus if p is such that $p^m \ll (\mathrm{N}_m(n, \mathcal{F}))^{-1}$, we have that $\mathrm{ex}(G_{n,p}, \mathcal{F}) < m$ asymptotically almost surely. □

Counting Cycle-free Graphs

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If $m \geq n^{1+1/(2\ell-1)}(\log n)^2$, then

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The proof used the method of hypergraph containers and a balanced supersaturation result. This result is essentially best possible if $\mathrm{ex}(n, \{C_3, \ldots, C_{2\ell}\}) = \Theta(n^{1+1/\ell})$.

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If $p \geq n^{-(\ell-1)/(2\ell-1)}(\log n)^{\ell+1}$, *then a.a.s.*

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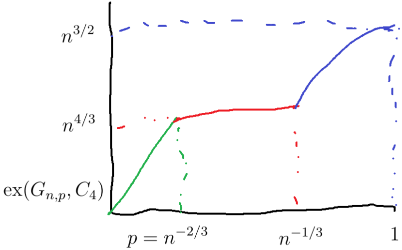
By using a more refined argument with containers one can get rid of this $\log n$ term.

Counting Cycle-free Graphs

Theorem (Füredi, 1991; Morris-Saxton, 2013)

$$\mathrm{ex}(G_{n,p}, C_4) = \begin{cases} (1+o(1))p\binom{n}{2} & n^{-1} \ll p \ll n^{-2/3}, \\ n^{4/3}(\log n)^{O(1)} & n^{-2/3} \le p \le n^{-1/3}(\log n)^4, \\ \Theta(p^{1/2}n^{3/2}) & n^{-1/3}(\log n)^4 \le p \le 1. \end{cases}$$



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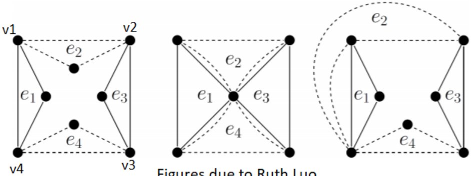
As before it is useful to define $\mathrm{N}^r_m(n, \mathcal{F})$ to be the number of $\mathcal{F}$-free r-graphs on n vertices with exactly m edges.

Berge Cycles and Girth

We say that F is a Berge C_ℓ if it has edges $e_1, \ldots, e_\ell$ and distinct vertices $v_1, \ldots, v_\ell$ with $v_i \in e_i \cap e_{i+1}$ for all i.



Figures due to Ruth Luo

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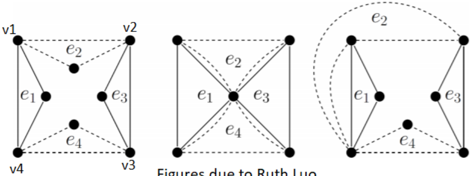

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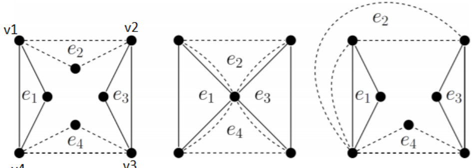

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Let $\mathcal{B}_\ell^r$ denote the set of r-uniform Berge C_ℓ's. A hypergraph H is said to have girth larger than ℓ if it is $\{\mathcal{B}_2^r, \ldots, \mathcal{B}_\ell^r\}$-free.

Berge Cycles and Girth

Theorem (S.-Verstraëte, 2020)

For $\ell, r \geq 3$ we have

$$\mathrm{N}^r_m(n, \{\mathcal{B}^r_2, \ldots, \mathcal{B}^r_\ell\}) \leq \mathrm{N}^2_m(n, \{C_3, \ldots, C_\ell\})^{r-1+\lceil \frac{r-2}{\ell-2} \rceil}.$$

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In particular, this allows us to lift the bounds of Morris-Saxton to hypergraphs. When $\ell = 3$ this gives tight bounds for all r:

Theorem (S.-Verstraëte, 2020)

For $p \geq n^{-r+3/2}(\log n)^{2r-3}$, we have a.a.s.

$$\mathrm{ex}(H^r_{n,p}, \{\mathcal{B}^r_2 \cup \mathcal{B}^r_3\}) = p^{\frac{1}{2r-3}} n^{2+o(1)},$$

and for significantly smaller values of p this equals $\Theta(pn^r)$.

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To illustrate the proof idea, we prove the weaker bound

$$\mathrm{N}^r_m(n, \{\mathcal{B}^r_2, \ldots, \mathcal{B}^r_\ell\}) \le \mathrm{N}^2_m(n, \{C_3, \ldots, C_\ell\})^{\binom{r}{2}}.$$

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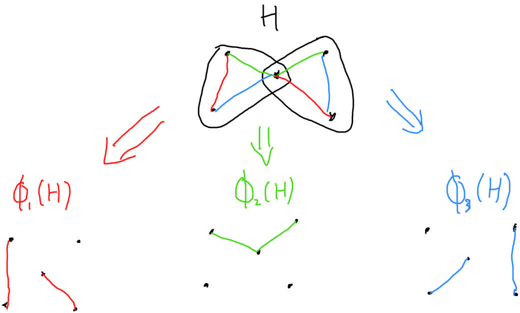
For any hypergraph H, go through each $e \in E(H)$ and order all of its $\binom{r}{2}$ pairs of vertices. Define the graph $\phi_i(H)$ by taking the ith pair from each hyperedge of H and adding it as an edge in $\phi_i(H)$.



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It's easy to show that if H has girth larger than ℓ and m edges, then so does $\phi_i(H)$. Thus the map

$$\phi(H) := (\phi_1(H), \ldots, \phi_{\binom{r}{2}}(H))$$

sends r-graphs with m edges and girth larger than ℓ to $\binom{r}{2}$ graphs with m edges and girth larger than ℓ.

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H is a hypergraph of girth larger than $\ell > 3$, $\phi_i(H)$ is the graph using the ith pair of each $e \in E(H)$,

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The key fact in proving the weaker result was that if H has large girth and we replace each hyperedge by a clique, then H is uniquely recoverable from this graph.

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The key fact in proving the weaker result was that if H has large girth and we replace each hyperedge by a clique, then H is uniquely recoverable from this graph. To get this stronger bound, we observe the stronger fact that we can replace each hyperedge with a graph K consisting of cycles of length at most ℓ all sharing a common edge and still be uniquely recoverable. □

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Theorem (S.-Verstraëte)

If $2 \leq \ell' \leq 4$, then

$$\mathrm{N}_m^r(n, \mathcal{B}_{\ell'}^r \cup \mathcal{B}_\ell^r) \leq 2^{cm} \cdot \mathrm{N}_m^2(n, C_\ell)^{\binom{r}{2}}.$$

Open Problems

Question

For $\ell \geq 3$, does there exist a constant c_ℓ such that

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Conjecture

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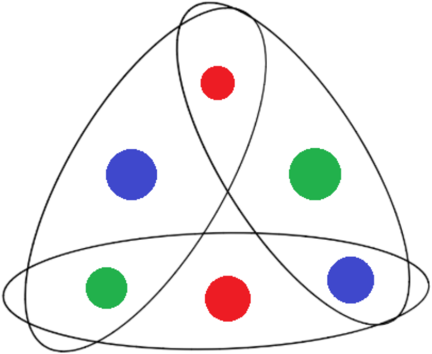
In particular, for $r = 3$ this would decrease the exponent from 3 to $2 + \frac{1}{\ell-2}$.

Open Problems

Define the r-uniform loose ℓ-cycle C_ℓ^r to be the r-graph with $e_1, \ldots, e_\ell$ and distinct vertices $v_1, \ldots, v_\ell$ such that $e_i \cap e_{i+1} = \{v_i\}$ and $e_i \cap e_j = \emptyset$ otherwise. For example, here is C_3^3.



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Theorem (Nie-S.-Verstraëte, 2020)

We have a.a.s.

$$\mathrm{ex}(H^3_{n,p}, C^3_3) = \begin{cases} (1+o(1))p\binom{n}{3} & n^{-1/3} \ll p \le n^{-3/2+o(1)}, \\ p^{1/3} n^{2+o(1)} & n^{-3/2+o(1)} \le p \le 1. \end{cases}$$

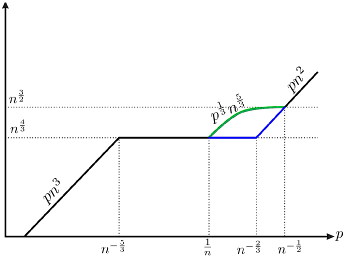
Theorem (Mubayi-Yepremyan, 2020)

For all $\ell \ge 2,\ r \ge 3$, we have a.a.s.

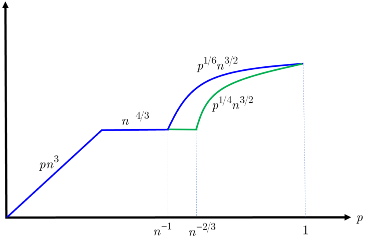
$$\mathrm{ex}(H^r_{n,p}, C^r_{2\ell}) \le \begin{cases} p^{\frac{1}{2\ell-1}} n^{1+\frac{r-1}{2\ell-1}+o(1)} & n^{-(r-2)+o(1)} \le p \le n^{-(r-2)+\frac{1}{2\ell-2}+o(1)} \\ pn^{r-1+o(1)} & n^{-(r-2)+\frac{1}{2\ell-2}+o(1)} \le p \le 1. \end{cases}$$

Open Problems

We have the following bounds for 3-uniform 4-cycles (with figures taken from Mubayi-Yepremyan and S.-Verstraëte, respectively):



$\mathbb{E}[\mathrm{ex}(H^3_{n,p}, C^3_4)]$



$\mathbb{E}[\mathrm{ex}(H^3_{n,p}, \{\mathcal{B}^3_2 \cup \mathcal{B}^3_3 \cup \mathcal{B}^3_4\})]$

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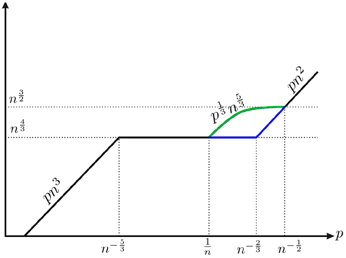

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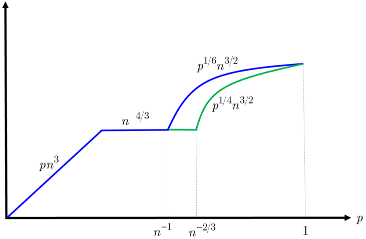

$\mathbb{E}[\mathrm{ex}(H^3_{n,p}, \{\mathcal{B}^3_2 \cup \mathcal{B}^3_3 \cup \mathcal{B}^3_4\})]$

If our previous conjecture is true, then we can improve the second upper bound from $p^{1/6}$ to $p^{1/5}$, but in any case we still have a gap.

Open Problems

The tight cycle T_ℓ^r is the hypergraph on $\{v_1, \ldots, v_\ell\}$ with all edges of the form $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\}$.

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Can one say anything about $\mathrm{ex}(H^r_{n,p}, T_\ell^r)$?

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Question

Can one say anything about $\mathrm{ex}(H_{n,p}^r, T_\ell^r)$?

This seems tricky because we don't even have good conjectures for $\mathrm{ex}(n, T_\ell^r)$.

Open Problems

One can extend the method for Berge cycles to Berge theta graphs to graphs which avoid theta graphs, so to get results in this case it suffices to have effective bounds for theta-free graphs.

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Problem

Determine tight bounds for counting theta-free graphs with m edges.

The End

Thank You!