Extremal Problems for Random Objects

Sam Spiro, Rutgers University

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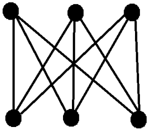
Theorem (Mantel 1907)

$$\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor .$$



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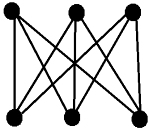

Theorem (Erdős-Stone 1946)

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}.$$

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The lower bound is tight when $p = 1$. The upper bound is tight if p is "small."

$$\frac{1}{2}p\binom{n}{2} \lesssim \mathrm{ex}(G_{n,p}, K_3) \lesssim p\binom{n}{2},$$

with the lower bound tight for $p = 1$ and the upper bound tight for $p \ll n^{-1/2}$.

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Theorem (Frankl-Rödl 1986)

Whp,

$$\mathrm{ex}(G_{n,p}, K_3) \sim \frac{1}{2}p\binom{n}{2} \qquad p \gg n^{-1/2}.$$

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Theorem (Conlon-Gowers, Schacht 2010)

Whp,

$$\mathrm{ex}(G_{n,p}, F) = p \cdot \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2} \qquad\qquad p \gg n^{-1/m_2(F)},$$

where $m_2(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subseteq F\}$.

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Conjecture

If F is a bipartite graph which is not a forest, then whp

$$\mathrm{ex}(G_{n,p}, F) = \begin{cases} \Theta(p \cdot \mathrm{ex}(n, F)) & p \gg n^{-1/m_2(F)}, \\ (1 + o(1))p\binom{n}{2} & p \ll n^{-1/m_2(F)}. \end{cases}$$

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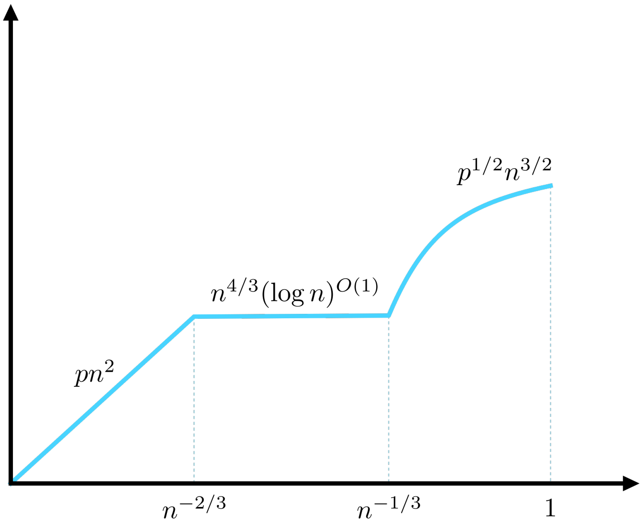
This conjecture turns out to be completely false!



Plot of $\mathsf{ex}(G_{n,p}, C_4)$ (Füredi 1991)

Conjecture (McKinley-S.)

If F is a graph with $\mathrm{ex}(n, F) = \Theta(n^\alpha)$ *for some* $\alpha \in (1, 2]$, *then whp*

$$\mathrm{ex}(G_{n,p}, F) = \max\{\Theta(p^{\alpha-1} n^\alpha), n^{2-1/m_2(F)} (\log n)^{O(1)}\},$$

provided $p \gg n^{-1/m_2(F)}$.

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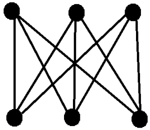
Theorem (Nie-S. 2023 (Informal))

This conjecture (essentially) implies Sidorenko's conjecture.

Theorem (Kővari-Sós-Turán 1954)

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$



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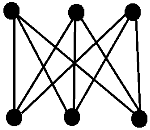

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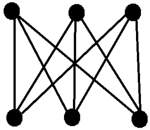

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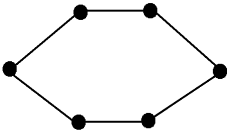
Moreover, this bound is tight whenever $\text{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$.

Theorem (Bondy-Simonovits 1974)

$$\mathrm{ex}(n, C_{2b}) = O(n^{1+1/b}).$$



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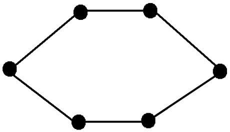

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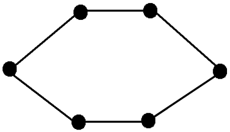

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Moreover, this is tight whenever $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2b}\}) = \Theta(n^{1+1/b})$.

Theorem (Jiang-Longbrake 2022)

If F satisfies "mild conditions", then

$$\mathrm{ex}(G_{n,p}, F) = O(p^{1-m_2^*(F)(2-\alpha)} n^{\alpha}) \text{ for } p \text{ large,}$$

where $m_2^*(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subsetneq F,\ e(F') \geq 2\}$.

Theorem (Faudree-Simonovits 1974)

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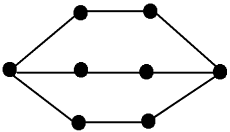

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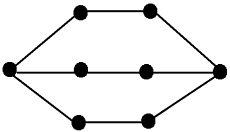

Theorem (Corsten-Tran 2021)

$$\mathrm{ex}(G_{n,p}, \theta_{a,b}) = O(p^{\frac{2}{ab}} n^{1+1/b}) \text{ for } p \text{ large.}$$

Note: our conjecture predicts $p^{\frac{1}{b}} n^{1+1/b}$.

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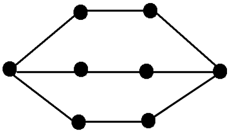

Theorem (McKinley-S. 2023)

For $a \geq 100$,

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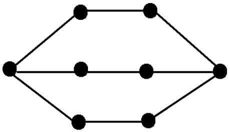

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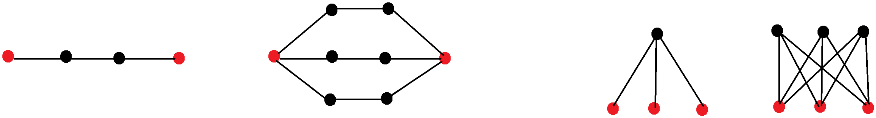
Moreover, this bound is tight whenever a is sufficiently large in terms of b.

Theorem (Bukh-Conlon 2015)

If T^ℓ is the "ℓth power of a balanced tree" and ℓ is sufficiently large, then

$$\mathrm{ex}(n, T^\ell) = \Omega(n^{2-\rho(T)}).$$



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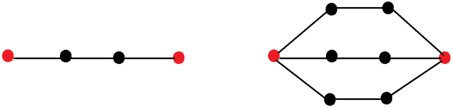

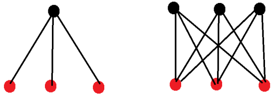

Theorem (S. 2022)

$$\mathrm{ex}(G_{n,p}, T^\ell) = \Omega(p^{1-\rho(T)} n^{2-\rho(T)}),$$

provided ℓ is sufficiently large.

Upper Bound Techniques

Upper Bound Techniques

Proof.

Containers. □

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Containers. □

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Hypergraph containers. □

Lower Bound Techniques

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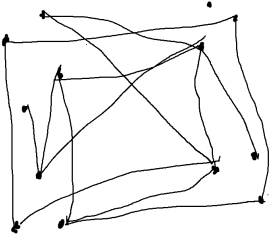

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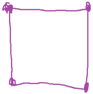
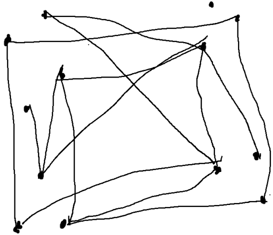


Lower Bound Techniques

$G_{n,p} =$



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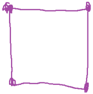

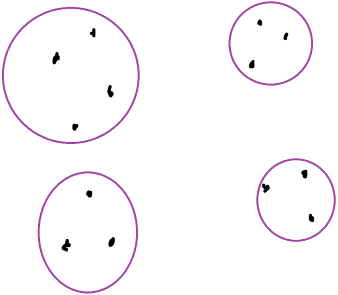

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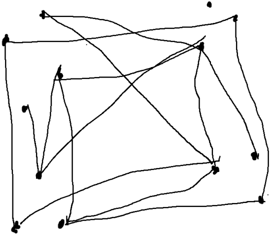

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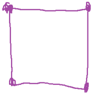
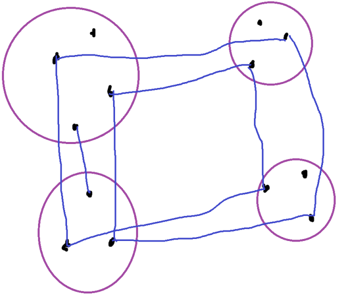
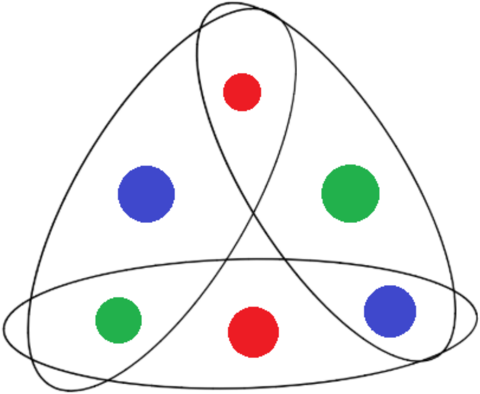




Hypergraphs



Theorem (S.-Verstraëte 2021)

Let $K^r_{s_1,\ldots,s_r}$ denote the complete r-partite r-graph with parts of sizes $s_1,\ldots,s_r$. There exist constants $\beta_1,\beta_2,\beta_3,\gamma$ depending on $s_1,\ldots,s_r$ such that, for s_r sufficiently large in terms of $s_1,\ldots,s_{r-1}$, we have whp

$$\mathrm{ex}(G^r_{n,p},K^r_{s_1,\ldots,s_r})=\begin{cases}\Theta\left(pn^r\right) & n^{-r}\ll p\le n^{-\beta_1},\\ n^{r-\beta_1+o(1)} & n^{-\beta_1}\le p\le n^{-\beta_2}(\log n)^{\gamma},\\ \Theta(p^{1-\beta_3}n^{r-\beta_3}) & n^{-\beta_2}(\log n)^{\gamma}\le p\le 1.\end{cases}$$

Question

Does the McKinley-Spiro conjecture extend to hypergraphs?

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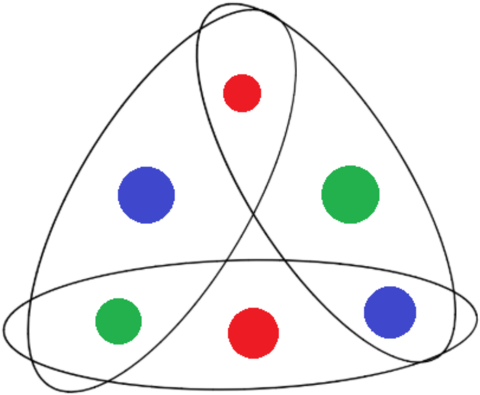
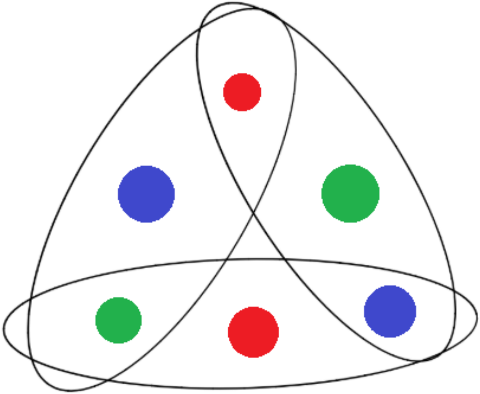
Theorem (Nie-S. 2023 (Informal))

Many hypergraphs fail to have a flat middle range. More precisely, any hypergraph which isn't Sidorenko fails to have a flat middle range.



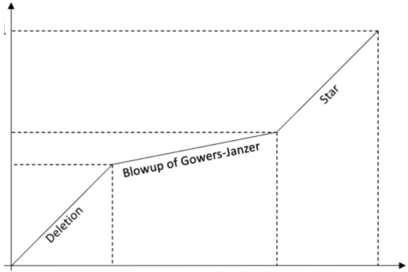


We define the *loose cycle* C_ℓ^r to be the r-uniform hypergraph obtained by inserting $r-2$ distinct vertices into each edge of the graph cycle C_ℓ.

Theorem (Nie-S.-Verstaëte 2020; Nie 2023)

For $r \geq 3$, if $p \gg n^{-r+3/2}$ then whp

$$\mathrm{ex}(G^r_{n,p}, C^r_3) = \max\{p^{\frac{1}{2r-3}} n^{2+o(1)}, pn^{r-1+o(1)}\}.$$



Picture due to Jiaxi Nie.

Theorem (Mubayi-Yepremyan 2020; Nie 2023)

For $r \geq 4$, if $p \gg n^{-r+1+\frac{1}{2\ell-1}}$ then whp

$$\mathrm{ex}(G^r_{n,p}, C^r_{2\ell}) = \max\{n^{1+\frac{1}{2\ell-1}}, pn^{r-1}\}.$$

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It's suspected that this continues to hold for $r = 3$, but there is a gap for medium values of p.

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It's suspected that this continues to hold for $r = 3$, but there is a gap for medium values of p.

Bounds also are known for Berge cycles, but the bounds are significantly weaker (S.-Verstraëte; Nie).

Theorem (Nie-S. 20XX (Informal))

If F is a graph and one has upper bounds for $\mathrm{ex}(G_{n,p}, F)$, then one can prove corresponding bounds for $\mathrm{ex}(G^r_{n,p}, F^{+r})$.

Here F^{+r} is the r-graph obtained by inserting $r-2$ new vertices inside each edge.

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Here F^{+r} is the r-graph obtained by inserting $r-2$ new vertices inside each edge.

Corollary

We have tight bounds for $\mathrm{ex}(G^r_{n,p}, K^{+r}_{s,t})$ if $r \geq s+2$.

Future Problems

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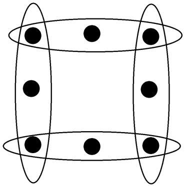
Prove tight bounds for the 3-uniform loose 4-cycle.



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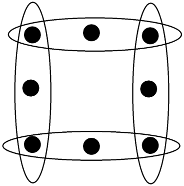

Problem

Prove tight bounds for subdivisions of complete bipartite graphs.