Random Turán Problems

Sam Spiro, Rutgers University

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Theorem (Erdős-Stone 1946)

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}.$$

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and with high probability

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The lower bound is tight when $p = 1$. The upper bound is tight if p is "small."

$$\frac{1}{2}p\binom{n}{2} \lesssim \mathrm{ex}(G_{n,p}, K_3) \lesssim p\binom{n}{2},$$

with the lower bound tight for $p = 1$ and the upper bound tight for $p \ll n^{-1/2}$.

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Theorem (Frankl-Rödl 1986)

Whp,

$$\mathrm{ex}(G_{n,p}, K_3) \sim \frac{1}{2}p\binom{n}{2} \qquad p \gg n^{-1/2}.$$

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Theorem (Conlon-Gowers, Schacht 2010)

Whp,

$$\mathrm{ex}(G_{n,p}, F) = p \cdot \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2} \qquad p \gg n^{-1/m_2(F)},$$

where $m_2(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subseteq F\}$.

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Conjecture

If F is a bipartite graph which is not a forest, then whp

$$\mathrm{ex}(G_{n,p}, F) = \begin{cases} \Theta(p \cdot \mathrm{ex}(n, F)) & p \gg n^{-1/m_2(F)}, \\ (1 + o(1))p\binom{n}{2} & p \ll n^{-1/m_2(F)}. \end{cases}$$

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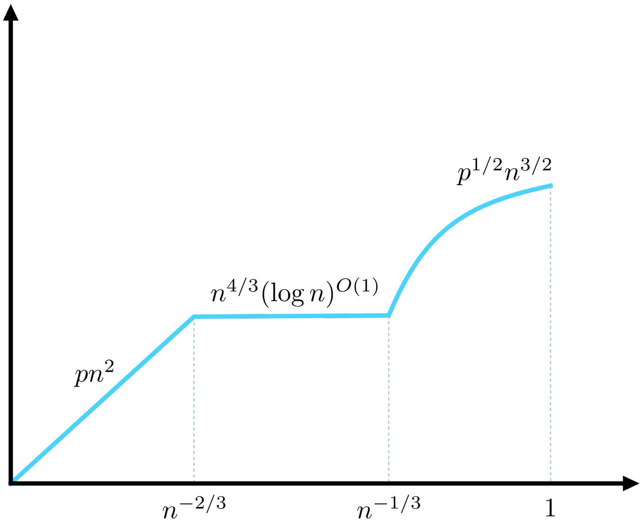
This conjecture turns out to be completely false!



Plot of $\mathsf{ex}(G_{n,p}, C_4)$ (Füredi 1991)

Conjecture (McKinley-S.)

If F is a graph with $\mathrm{ex}(n, F) = \Theta(n^\alpha)$ for some $\alpha \in (1, 2]$, then whp

$$\mathrm{ex}(G_{n,p}, F) = \max\{\Theta(p^{\alpha-1} n^\alpha), n^{2-1/m_2(F)} (\log n)^{O(1)}\},$$

provided $p \gg n^{-1/m_2(F)}$.

Theorem (Kővari-Sós-Turán 1954)

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$

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Moreover, this bound is tight whenever $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$.

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Moreover, this is tight whenever $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2b}\}) = \Theta(n^{1+1/b})$.

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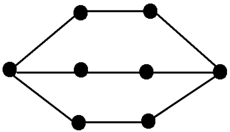

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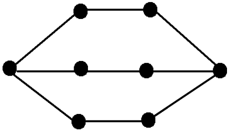

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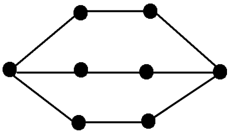

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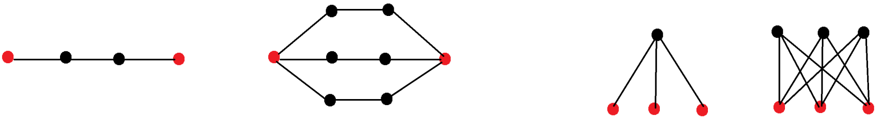
Moreover, this bound is tight whenever a is sufficiently large in terms of b.

Theorem (Bukh-Conlon 2015)

If T^ℓ is the "ℓth power of a balanced tree with density b/a", then $\mathrm{ex}(n, T^\ell) = \Omega(n^{2-a/b})$ *if ℓ is sufficiently large.*



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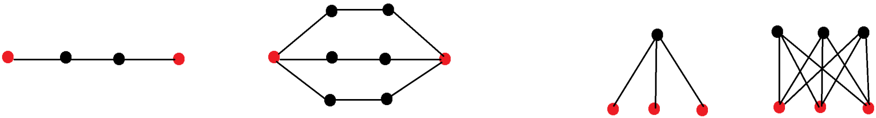


Theorem (S. 2022)

$$\mathrm{ex}(G_{n,p}, T^\ell) = \Omega(p^{1-a/b} n^{2-a/b}).$$

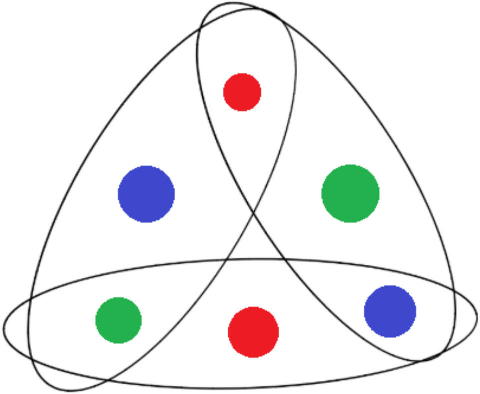
Theorem (Jiang-Longbrake 2022)

If F satisfies "mild conditions", then

$$\mathrm{ex}(G_{n,p}, F) = O(p^{1-m_2^*(F)(2-\alpha)} n^{\alpha}) \text{ for } p \text{ large},$$

where $m_2^*(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subsetneq F,\ e(F') \geq 2\}$.

Hypergraphs



Hypergraphs

Theorem (S.-Verstraëte 2021)

Let $K^r_{s_1,\ldots,s_r}$ denote the complete r-partite r-graph with parts of sizes $s_1,\ldots,s_r$. There exist constants $\beta_1,\beta_2,\beta_3,\gamma$ depending on $s_1,\ldots,s_r$ such that, for s_r sufficiently large in terms of $s_1,\ldots,s_{r-1}$, we have whp

$$\mathrm{ex}(G^r_{n,p},K^r_{s_1,\ldots,s_r})=\begin{cases}\Theta\left(pn^r\right) & n^{-r}\ll p\le n^{-\beta_1},\\ n^{r-\beta_1+o(1)} & n^{-\beta_1}\le p\le n^{-\beta_2}(\log n)^{\gamma},\\ \Theta(p^{1-\beta_3}n^{r-\beta_3}) & n^{-\beta_2}(\log n)^{\gamma}\le p\le 1.\end{cases}$$

Hypergraphs

Question

Does the McKinley-Spiro conjecture extend to hypergraphs?

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Theorem (Nie-S. 2023 (Informal))

Any hypergraph which is not Sidorenko fails to have a flat middle range.

Hypergraphs

We define the *loose cycle* C_ℓ^r to be the r-uniform hypergraph obtained by inserting $r-2$ distinct vertices into each edge of the graph cycle C_ℓ.

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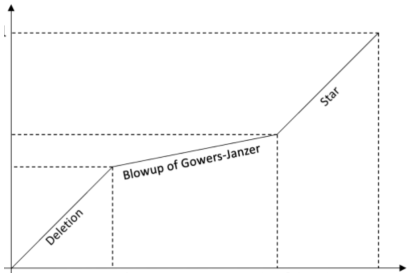
Theorem (Nie-S.-Verstaëte 2020; Nie 2023)

For $r \geq 3$, if $p \gg n^{-r+3/2}$ then whp

$$\mathrm{ex}(G_{n,p}^r, C_3^r) = \max\{p^{\frac{1}{2r-3}} n^{2+o(1)}, pn^{r-1+o(1)}\}.$$



Picture due to Jiaxi Nie.

Hypergraphs

Theorem (Mubayi-Yepremyan 2020; Nie 2023)

For $r \geq 4$, if $p \gg n^{-r+1+\frac{1}{2\ell-1}}$ then whp

$$\mathrm{ex}(G^r_{n,p}, C^r_{2\ell}) = \max\{n^{1+\frac{1}{2\ell-1}}, pn^{r-1}\}.$$

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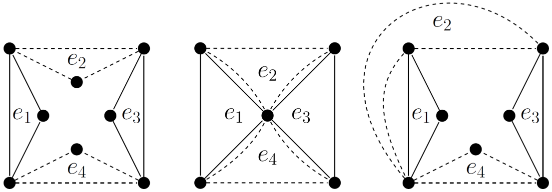
It's suspected that this continues to hold for $r = 3$, but there is a gap for medium values of p.

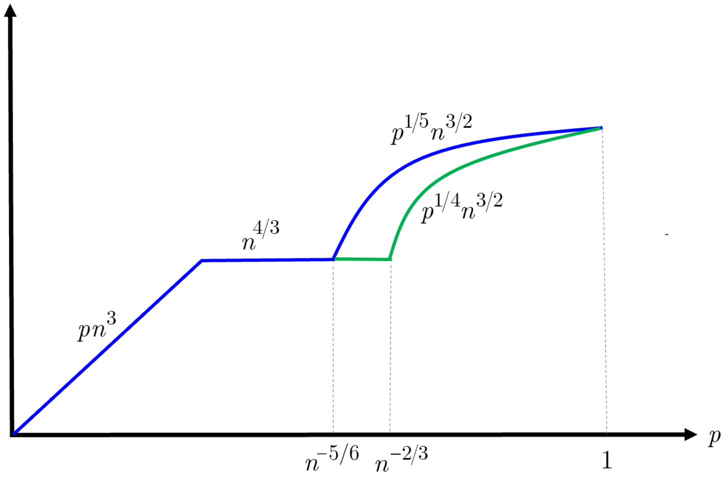
Hypergraphs

We say that F is a Berge C_ℓ if it has edges $e_1, \ldots, e_\ell$ and distinct vertices $v_1, \ldots, v_\ell$ with $v_i \in e_i \cap e_{i+1}$ for all i.



Hypergraphs

Plot of $\mathrm{ex}(G^3_{n,p}, \mathcal{B}^3(C_4))$



S.-Verstraëte 2021; Nie 2023

Hypergraphs

Theorem (Nie-S. 20XX (Informal))

If F is a graph and one has upper bounds for $\mathrm{ex}(G_{n,p}, F)$, then one can prove corresponding bounds for $\mathrm{ex}(G_{n,p}^r, \mathrm{Ex}^r(F))$ and $\mathrm{ex}(G_{n,p}^r, \mathcal{B}^r(F))$.

Upper Bound Techniques

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Proof.

Containers. □

Upper Bound Techniques

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Hypergraph containers. □

Lower Bound Techniques

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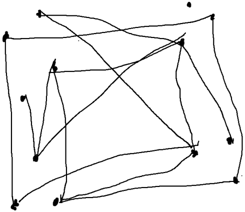
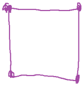
$G_{n,\ell} =$ 

$ex(P_n, F) =$ 

Lower Bound Techniques

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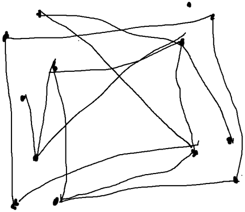

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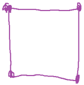

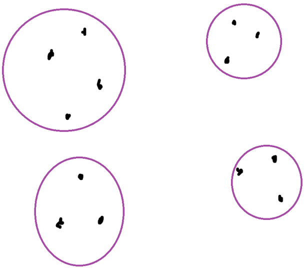

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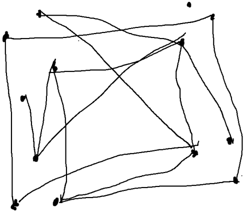

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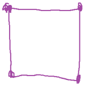

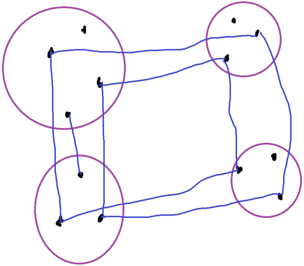

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$$t_F(H) = \frac{\#\text{homs } F \to H}{v(H)^{v(F)}}.$$

We say that a hypergraph F is *Sidorenko* if for all r-graphs H, we have

$$t_F(H) \geq t_{K_r^r}(H)^{e(F)}.$$

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A graph F is Sidorenko if and only if F is bipartite.

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Theorem (Conlon-Lee-Sidorenko 2023)

If F is an r-graph which is not Sidorenko, then there exists $\epsilon = \epsilon(F) > 0$ such that

$$\mathrm{ex}(n, F) = \Omega(n^{r - \frac{v(F)-r}{e(F)-1} + \epsilon}).$$

Sidorenko's Conjecture

For an r-graph F, define

$$s(F) := \sup\{s : \exists H \neq \emptyset,\ t_F(H) = t_{K_r^r}(H)^{s+e(F)}\}.$$

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Theorem (Nie-S. 2023)

If F is an r-graph with $e(F) \geq 2$ and $\frac{v(F)-r}{e(F)-1} < r$, then for any $p = p(n) \geq n^{-\frac{v(F)-r}{e(F)-1}}$, we have whp

$$\mathrm{ex}(G_{n,p}^r, F) \geq n^{r-\frac{v(F)-r}{e(F)-1}-o(1)}\left(pn^{\frac{v(F)-r}{e(F)-1}}\right)^{\frac{s(F)}{e(F)-1+s(F)}}.$$

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Lemma

If F is an r-graph such that there exists an r-graph H with $t_{K_r^r}(H) = \alpha$ and $t_F(H) = \beta$, then for all r-graphs G and integers $N \geq 1$ we have

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Given two r-graphs H, H', we define the *tensor product* $H \otimes H'$ to be r-graph on $V(H) \times V(H')$ where $((x_1, y_1), \ldots, (x_r, y_r)) \in E(H \otimes H')$ if and only if $(x_1, \ldots, x_r) \in E(H)$ and $(y_1, \ldots, y_r) \in E(H')$.

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Fact: for any r-graphs F, H and $N \geq 1$, we have

$$t_F(H^{\otimes N}) = t_F(H)^N.$$

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$$\mathbb{E}[e(G')] = t_{K_r^r}(H^{\otimes N}) \cdot e(G)$$

Proof of Main Theorem

Lemma

If F is an r-graph such that there exists an r-graph H with $t_{K_r^r}(H) = \alpha$ and $t_F(H) = \beta$, then for all r-graphs G and integers $N \geq 1$ we have

$$\text{ex}(G, F) \geq \alpha^N e(G) - \beta^N \mathcal{N}_F(G).$$

Let $\phi : V(G) \to V(H^{\otimes N})$ be chosen uniformly at random, and define $G' \subseteq G$ by keeping the edges which map bijectively to edges.

$$\mathbb{E}[e(G')] = t_{K_r^r}(H^{\otimes N}) \cdot e(G) = \alpha^N \cdot e(G)$$

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One gets the result by deleting an edge from each copy of F in G'. □

Further Results

$$s(F) := \sup\{s : \exists H \neq \emptyset,\ t_F(H) = t_{K_r^r}(H)^{s+e(F)}\}.$$

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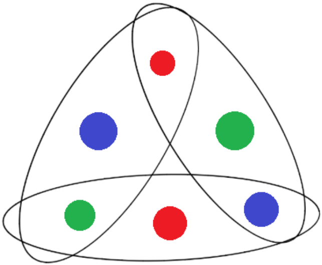
Given a k-graph F, its *expansion* $\mathrm{Ex}^r(F)$ is defined by inserting $r-k$ new vertices into each edge of F.



Further Results

Theorem (Nie-S. 2023)

If F is a k-graph which contains K_{k+1}^{k} as a subgraph, then

$$s(\mathrm{Ex}^{r}(F)) \geq \frac{1}{r-k}.$$

In particular, $\mathrm{Ex}^{r}(F)$ is not Sidorenko.

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Theorem (Nie-S. 2023)

$$s(\mathrm{Ex}^r(F)) \leq \frac{v(F)-k}{v(F)-k+(r-k)(s(F)+e(F)-1)} \cdot s(F).$$

In particular, expansions of Sidorenko hypergraphs are Sidorenko.

Open Problems

Theorem (Nie-S. 2023)

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Conjecture

For every bipartite graph F, there exists an $r \geq 2$ such that $\mathrm{Ex}^r(F)$ is Sidorenko.

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For every bipartite graph F, there exists an $r \geq 2$ such that $\mathrm{Ex}^r(F)$ is Sidorenko.

Question

Is it true that F is Sidorenko if and only if there exists an expansion $\mathrm{Ex}^r(F)$ which is Sidorenko?

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For every bipartite graph F, there exists an $r \geq 2$ such that $\mathrm{Ex}^r(F)$ is Sidorenko.

Question

Is it true that F is Sidorenko if and only if there exists an expansion $\mathrm{Ex}^r(F)$ which is Sidorenko? In particular, are all expansions of non-bipartite graphs not Sidorenko?

Open Problems

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Determine $s(C^r_{2\ell+1})$.

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Our best guess is

$$s(C^r_{2\ell+1}) = \frac{\ell}{(r-1)\ell - 1}.$$