Relative Turán Numbers of Hypergraphs

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Joint with Jiaxi Nie and Jacques Verstraëte

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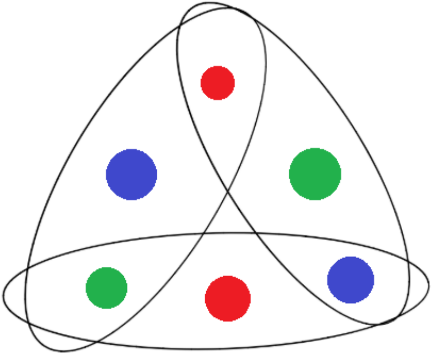

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Theorem (Erdős-Stone-Simonovits, 1946)

If F is a graph with $\chi(F) = k$, then

$$\mathrm{ex}(n, F) = \left(\frac{k-2}{k-1} + o(1) \right) \binom{n}{2}.$$

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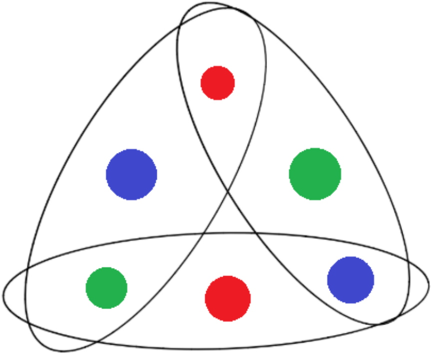
Mantel's theorem determines $\mathrm{ex}(n, C_3)$; what happens for triangle-free *hypergraphs*? Define the loose ℓ-cycle C_ℓ^r be the r-graph with $e_1, \ldots, e_\ell$ and distinct vertices $v_1, \ldots, v_\ell$ such that $e_i \cap e_{i+1} = \{v_i\}$ and $e_i \cap e_j = \emptyset$ otherwise. For example, here is C_3^3.



Turán Numbers

Theorem (Frankl-Füredi, 1987)

For $r \geq 3$ and n sufficiently large,

$$\mathrm{ex}(n, C_3^r) = \binom{n-1}{r-1},$$

with the extremal example being the star $S_{n,r}$ which has all r-sets containing a common vertex.

Relative Turán Numbers

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so morally speaking the relative Turán problem is the same for H and $m \cdot H$ despite their number of vertices being incomparable.

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In particular, the best general bound we could hope to prove for graphs is

$$\mathrm{ex}(H, \mathcal{F}) = \Omega(\mathrm{ex}(\Delta, \mathcal{F}) \Delta^{-2}) \cdot e(H).$$

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Conjecture (Foucaud-Krivelevich-Perarnau, 2014)

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Somewhat surprisingly they were able to prove these bounds despite us not knowing what $\mathrm{ex}(n, \mathcal{F})$ is for many of these cases.

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For any 3-graph H with maximum degree at most Δ, we have

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To find a large triangle-free subgraph of H, we will use a triangle-free 3-graph J as a "template."

Random Homomorphisms and C_3^3

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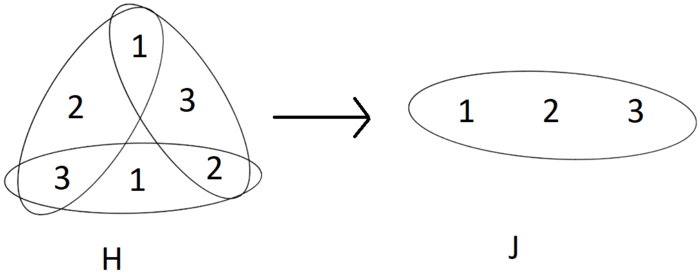

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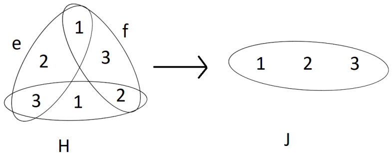

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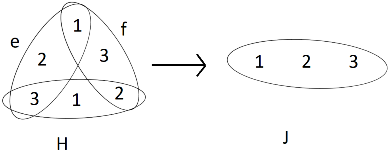

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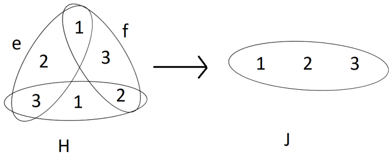

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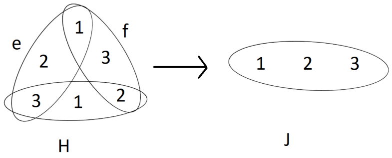

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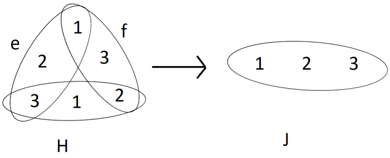
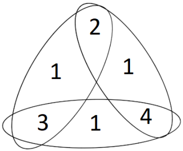


Redefine $H' \subseteq H$ to have the edges $e \in E(H)$ such that (1) $\chi(e) \in E(J)$, and (2) for any $f \in E(H)$ with $|f \cap e| = 1$ we have $\chi(f) \neq \chi(e)$.This solves the previous issue, but there are still issues that can happen. For example, if J is the star 3-graph $S_{n,3}$ with common element 1, then a triangle in H will survive if it's given the following assignment



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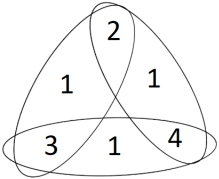

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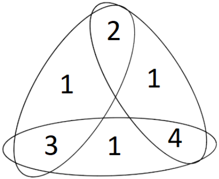

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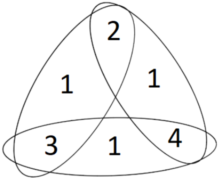

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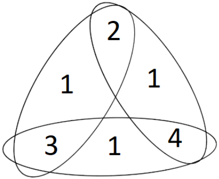

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Theorem (Ruzsa-Szemerédi, 1978)

There exists a t-vertex 3-graph R_t with $t^{2-o(1)}$ edges which is triangle-free and which is linear.

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Assume $e_1, e_2, e_3 \in H'$ forms a triangle with $e_i \cap e_j = \{x_{ij}\}$. Note that $\chi(x_{ij}) \in \chi(e_i) \cap \chi(e_j)$, and that these vertices must be distinct (since $\chi(x_{12}) = \chi(x_{13})$ implies $|\chi(e_1)| < 3$). Further, $|\chi(e_i) \cap \chi(e_j)| \neq 2, 3$. Thus $\chi(e_i) \cap \chi(e_j) = \{\chi(x_{ij})\}$. Thus $\chi(e_1), \chi(e_2), \chi(e_3)$ is a C_3^3 in R_t, a contradiction.

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Let $J = R_t$ and $\chi : V(H) \to V(J)$ be chosen randomly. Define $H' \subseteq H$ to have the edges $e \in E(H)$ such that (1) $\chi(e) \in E(J)$, and (2) for any $f \in E(H)$ with $|f \cap e| = 1$ we have $\chi(f) \neq \chi(e)$.

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Linearity of expectation then gives $\mathbb{E}[e(H')] \geq \Delta^{-1/2-o(1)} e(H)$. $\square$

Random Homomorphisms and C_3^3

Theorem (Nie-S.-Verstraëte, 2020)

If H is an r-graph with maximum degree Δ, then

$$\mathrm{ex}(H, C_3^r) \geq \Delta^{-\frac{r-2}{r-1}-o(1)} \cdot e(H).$$

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In particular, the worst host is *not* a clique.

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A similar approach can be made to work for other $\mathcal{F}$. For example, let $K^3_{2,2,s}$ denote the complete 3-partite 3-graph with parts of sizes 2, 2, and s. If s is sufficiently large it is known that

$$\mathrm{ex}(n, K^3_{2,2,s}) = \Theta(n^{3-1/4}) = O(\Delta^{-1/8}) \cdot e(K^3_n).$$

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There exists a 3-graph H with maximum degree at most $\Delta \to \infty$ such that

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The proof of the lower bound requires two cases: one where the host H has small codegrees and one where it has high codegrees.

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Codegrees and $K^3_{2,2,s}$

Lemma

If H is a 3-graph with maximum codegree at most D and $\mathrm{ex}(n, K^3_{2,2,s}) = \Theta(n^{3-1/4})$, *then*

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This will give the correct answer of $\Delta^{-1/6}e(H)$ when $D \leq \Delta^{2/3}$, but we need a new approach for hosts with large codegrees.

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Lemma

If H is 3-partite on $V_1 \cup V_2 \cup V_3$ such that every pair in $V_1 \cup V_2$ has codegree 0 or D, then

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Roughly take G to be the graph induced by $V_1 \cup V_2$, find $G' \subseteq G$ which is C_4-free (using Perarnau-Reed), and then lift this to a subgraph in H.

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This gives $\mathrm{ex}(H, K^3_{2,2,s}) \geq \Delta^{-1/6-o(1)} e(H)$, and further shows that if this is sharp the host must have maximum codegree about $\Delta^{2/3}$.

Codegrees and $K^3_{2,2,s}$

Lemma

$$\mathrm{ex}(K^3_{n,n,n^2}, K^3_{2,2,s}) = O(\Delta^{-1/6}) \cdot e(K^3_{n,n,n^2}).$$

Other Results: Cycles

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Theorem (S.-Verstraëte, 2020+)

Let $\ell \geq 3$. If H is a 3-graph with maximum degree at most $\Delta \to \infty$, then

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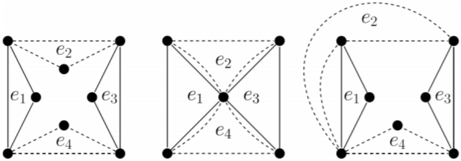
The upper bound uses a random host and results of Mubayi and Yepremyan.

Other Results: Cycles

We say that F is a Berge C_ℓ if it has edges $e_1, \ldots, e_\ell$ and distinct vertices $v_1, \ldots, v_\ell$ with $v_i \in e_i \cap e_{i+1}$ for all i. Let $\mathcal{B}_\ell^r$ denote the set of r-uniform Berge C_ℓ's.



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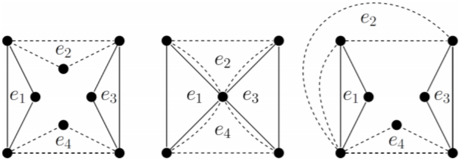

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Proposition

If $r > \ell$ then $\mathrm{ex}(H, \mathcal{B}_\ell^r) = \Omega(\Delta^{-1})e(H)$, and this is best possible.

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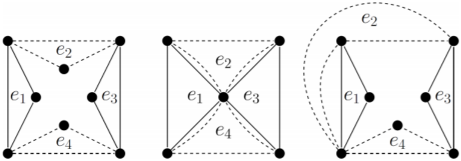

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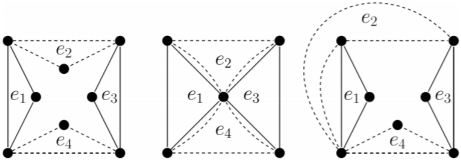

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If H is a 3-graph with maximum degree at most $\Delta \to \infty$, then

$$\mathrm{ex}(H, \mathcal{B}_3^3) \geq \Delta^{-1/2-o(1)} \cdot e(H),$$
$$\mathrm{ex}(H, \mathcal{B}_4^3) \geq \Delta^{-3/4-o(1)} \cdot e(H),$$
$$\mathrm{ex}(H, \mathcal{B}_5^3) \geq \Delta^{-3/4-o(1)} \cdot e(H).$$

If H is a 4-graph with maximum degree at most $\Delta \to \infty$, then

$$\mathrm{ex}(H, \mathcal{B}_4^4) \geq \Delta^{-5/6-o(1)} \cdot e(H).$$

Moreover, all of these bounds are tight up to a factor of $o(1)$.

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Theorem (S.-Verstraëte, 2020+)

If s is sufficiently large, then a.a.s.

$$\mathrm{ex}(H^3_{n,p}, K^3_{2,2,s}) = \begin{cases} \Theta(pn^3) & n^{-3+o(1)}\log n \le p \le n^{-\frac{-s-1}{4s-1}}, \\ n^{\frac{11s-4}{4s-1}+o(1)} & n^{\frac{-s-1}{4s-1}} \le p \le n^{\frac{-5}{12s-3}}, \\ p^{3/4}n^{3-1/4+o(1)} & n^{\frac{-5}{12s-3}} \le p. \end{cases}$$

Random Hosts

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If $\ell \geq 3$ *and* $\mathrm{ex}(n, \bigcup_{\ell'=2}^{\ell} \mathcal{B}_{\ell'}^3) \geq n^{1+1/\lfloor \ell/2 \rfloor - o(1)}$, *then a.a.s.*

$$\mathrm{ex}(H_{n,p}^3, \{\mathcal{B}_2^3, \ldots, \mathcal{B}_\ell^3\}) \leq p^{\frac{1}{3\lfloor \ell/2 \rfloor}} n^{1+1/\lfloor \ell/2 \rfloor + o(1)} \text{ for } p \geq n^{-3+\frac{\lfloor \ell/2 \rfloor}{\ell-1}},$$

$$\mathrm{ex}(H_{n,p}^3, \{\mathcal{B}_2^3, \ldots, \mathcal{B}_\ell^3\}) \geq p^{\frac{1}{2\lfloor \ell/2 \rfloor}} n^{1+1/\lfloor \ell/2 \rfloor - o(1)} \text{ for } p \geq n^{-2+\frac{\lfloor \ell/2 \rfloor}{\ell-1}}.$$

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Note that the girth problem is trivial for general hosts since sunflowers give $\mathrm{ex}(H, \mathcal{B}_2^r) = O(\Delta^{-1})e(H)$.

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We have a.a.s.

$$\mathrm{ex}(H^r_{n,p}, \{\mathcal{B}^r_2,\ \mathcal{B}^r_3\}) = \begin{cases} \Theta(pn^r) & n^{-3+o(1)} \le p \le n^{-r+3/2}, \\ p^{\frac{1}{2r-3}} n^{2+o(1)} & n^{-r+3/2} \le p. \end{cases}$$

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The same lower bound holds for forbidding $\mathcal{B}^r_3$ or C^r_3, but we do not have tight upper bounds when $r > 3$.

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- Obtain tighter bounds for $\mathrm{ex}(H_{n,p}^r, \bigcup_{\ell' \leq \ell} \mathcal{B}_\ell^r)$.
- Prove bounds for your favorite family of hypergraphs $\mathcal{F}$.

How'd This Go?

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Link also on my website.

The End

Thank You!