Odd Cycle Saturation Games

Sam Spiro, UC San Diego.

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The $\mathcal{F}$-saturation game

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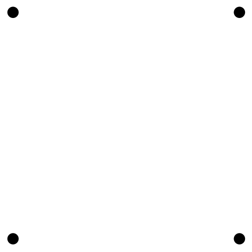
When the game ends, Max gets a point for every edge in G at the end of the game and Mini loses a point for every edge in G. Thus Max wants the game to last as long as possible, while Mini wants the game to end as quickly as possible.

The $\mathcal{F}$-saturation game

Example: the $\{C_3\}$-saturation game. Max goes first.



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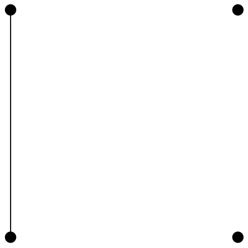

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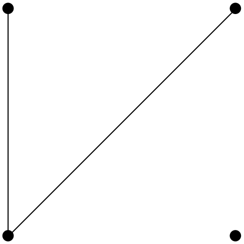

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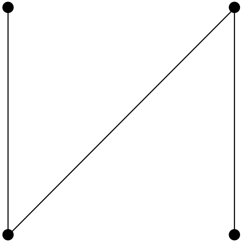

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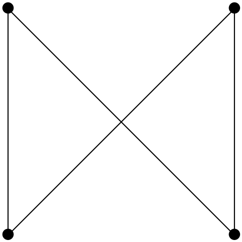

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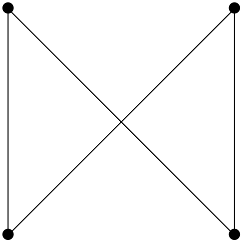

The graph is now C_3-saturated so the game ends. Max gets 4 points (the best he could possibly do) and Mini loses 4 points.

The $\mathcal{F}$-game saturation number

Let $\mathrm{sat}_g(\mathcal{F}; n)$ denote the number of edges in G at the end of the $\mathcal{F}$-saturation game when both players play optimally. The goal is to find this value, which is known as the $\mathcal{F}$-game saturation number.

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Example

$$n - 1 \leq \mathrm{sat}_g(\{C_3\}; n) \leq \lfloor \frac{1}{4} n^2 \rfloor.$$

The $\mathcal{F}$-game saturation number

Theorem (Furedi-Reimer-Sersess, 1992)

$$\mathrm{sat}_g(\{C_3\}; n) \geq \frac{1}{2} n \log n + o(n \log n).$$

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These are the only known bounds for the triangle-free game. Our goal is to establish a lower bound for a related game, namely the $\{C_3, C_5\}$-saturation game. Key idea: Max can force the graph to be bipartite throughout this game.

The $\{C_3, C_5\}$-saturation game

In general, let X^t denote X after t edges have been added in the game, e.g. G^t denotes the graph after t edges have been played, e^t denotes the edge added at time t, etc.

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Identify two adjacent vertices $u \in U^t$, $v \in V^t$. Let $U_b^t = U^t \setminus N(v)^t$ and $V_b^t = V^t \setminus N(u)^t$ (the bad vertices).

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How can Max play so that he can achieve this?

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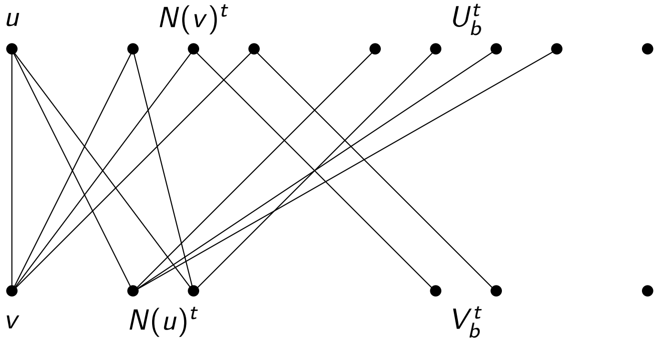
Inductively assume that Max plays so G^{t-2} satisfies (1*) and (2*).



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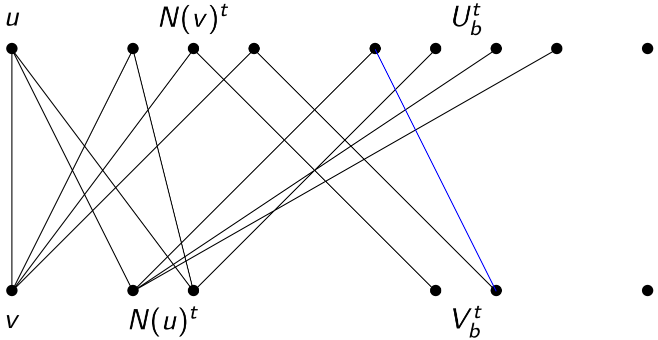

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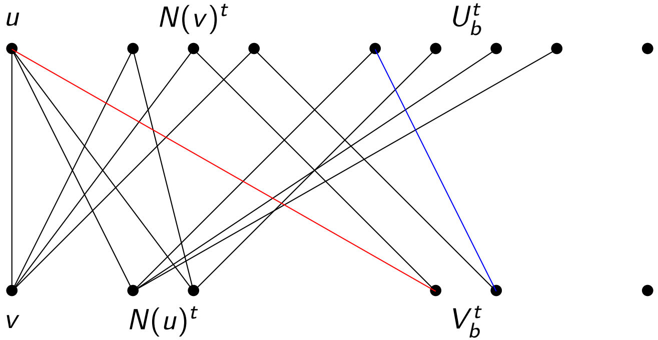

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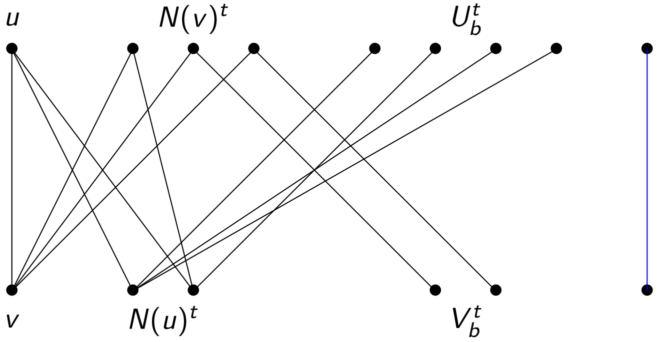

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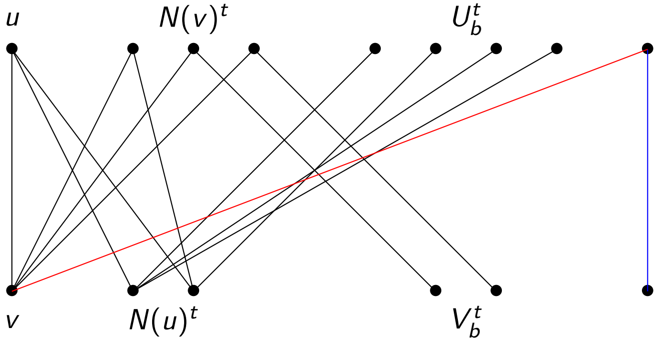

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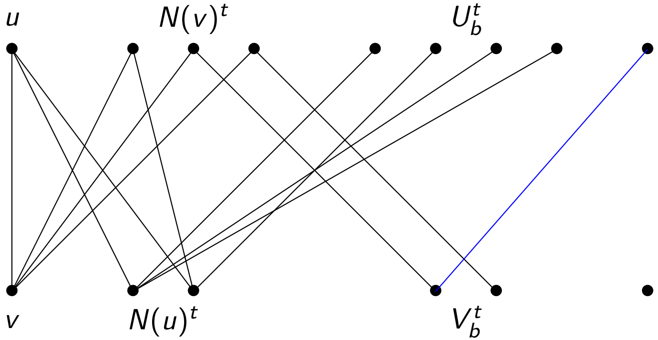

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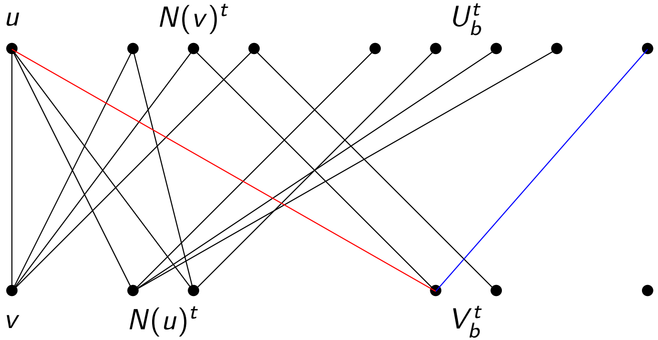

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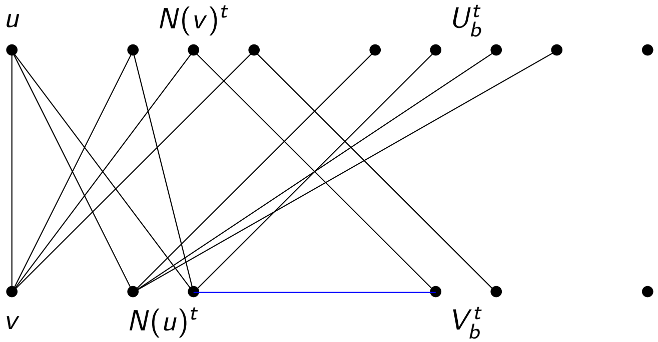

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Let t be such that G^t satisfies (1) and (2*). Then U^{t+1} and V^{t+1} are independent sets for any valid choice of e^{t+1} in the $\{C_3, C_5\}$-saturation game for $k \geq 2$.*

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Given this lemma, Mini can only do Internal, Outside, and Add to U/V moves, so Max can indeed play so that (1*) and (2*) are maintained.

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If Mini does an Internal or Outside move then Max acts as he did before, and with this b_U^t, b_V^t don't increase. However, Max has to be more careful when Mini plays an Add to U move.

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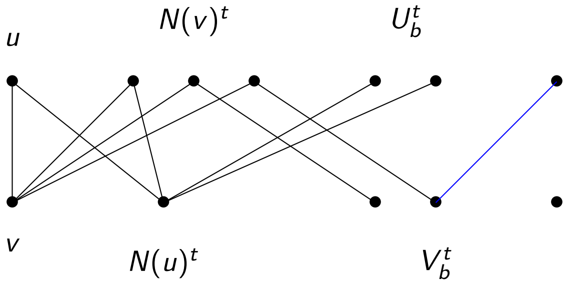

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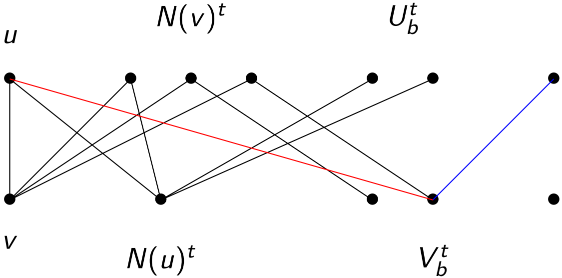

$$b_U^{t+2} = |V_b^{t+2}| + (|U^{t+2}| - 2|V^{t+2}|) = 0,$$
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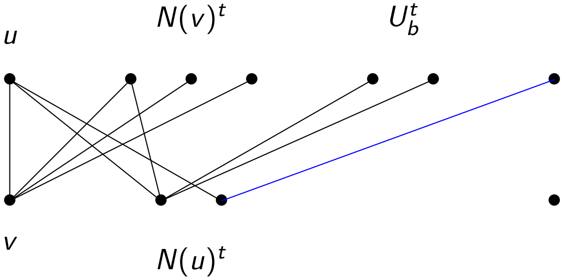
Case 2: $|U^{t+1}| > 2|V^{t+1}|$.



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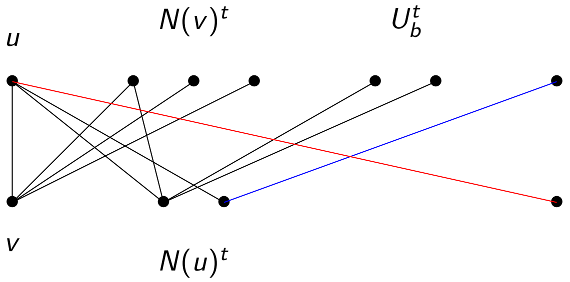

$$b_U^{t+2} = |V_b^{t+2}| + (|U^{t+2}| - 2|V^{t+2}|) = -1,$$
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Improving the constant

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Essentially one uses the same strategy as before but with a stronger induction. Namely, Max maintains the following.

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The main idea is that (4*) guarantees that one of $b_U^t, b_V^t \leq -1$, and hence one of the sets U^t, V^t can afford to have its structure disrupted.

The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

This same proof holds for any set of odd cycles $\mathcal{C}$ with $C_3, C_5 \in \mathcal{C}$.

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Idea for the lower bound: call a vertex bad if it's roughly distance k away from u or v (as opposed to those that simply aren't adjacent to u/v).

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Idea for the lower bound: call a vertex bad if it's roughly distance k away from u or v (as opposed to those that simply aren't adjacent to u/v). By being more careful in the previous argument, and by making a slight tweak to the strategy, one can replace the $\frac{3}{2}$ we had before with $\gamma_k \to 1$.

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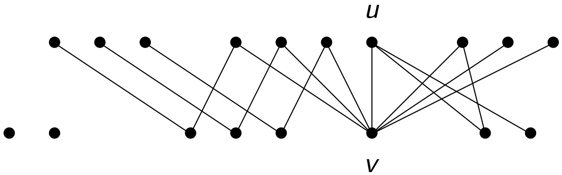
Path Growing Phase 3:



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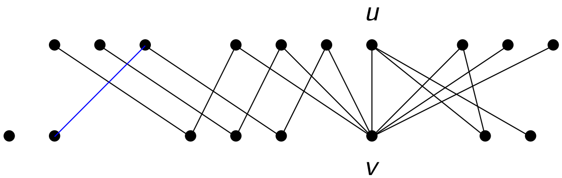

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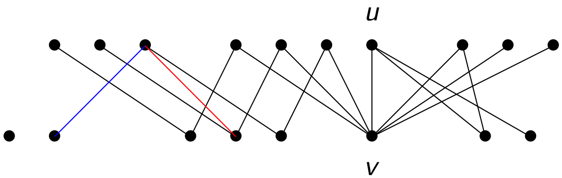

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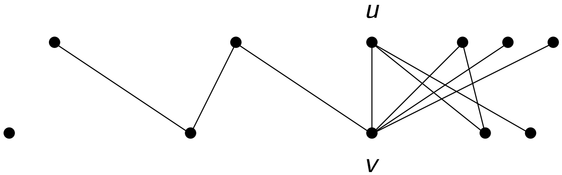

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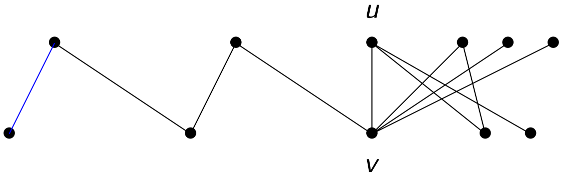

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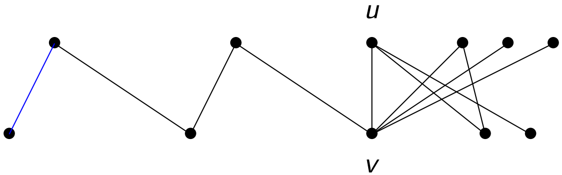

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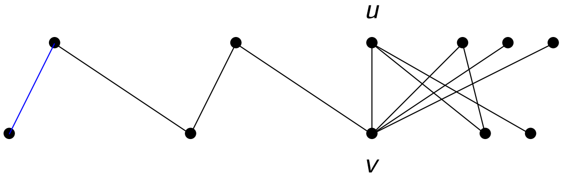

Every time Max destroys paths, $|V^t|$ increases while $|U^t|$ stays the same. Thus eventually either $|V^t|$ becomes much larger than $|U^t|$ (in which case Mini maintains this), or Mini succeeds in making many long paths (which eventually she'll connect to form C_{2k+1}'s).

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For $k \geq 4$,

$$\left(\frac{1}{4} - \frac{1}{5k^2}\right) n^2 + o(n^2) \leq \mathrm{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left(\frac{1}{4} - \frac{1}{20^6 k^4}\right) n^2 + o(n^2).$$

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Conjecture

For all $k \geq 1$ there exists a $c_k > 0$ such that

$$\mathrm{sat}(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left(\frac{1}{4} - c_k\right) n^2 + o(n^2).$$

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Conjecture

For all $k \geq 2$ and n sufficiently large,

$$\mathrm{sat}_g(\{C_3, \ldots, C_{2k-1}\}; n) \leq \mathrm{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n).$$

The $(\mathcal{C}_\infty \setminus \{C_3\})$-saturation game

Let $\mathcal{C}_\infty = \{C_3, C_5, C_7, \ldots\}$. We wish to consider the $(\mathcal{C}_\infty \setminus \{C_3\})$-saturation game.

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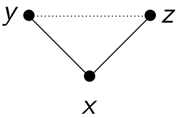
Key idea: Mini can play so that almost every edge of G^t lies in a triangle.

The $(\mathcal{C}_\infty \setminus \{C_3\})$-saturation game

Lemma

If xy and xz are not in triangles, then yz is a legal move in the $(\mathcal{C}_\infty \setminus \{C_3\})$-saturation game.



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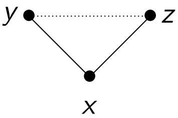

By doing this repeatedly, Mini can guarantee that "most" edges are in triangles.

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If G is a graph where "most" edges are in triangles and G contains no C_k with $k \geq 5$ odd, then G contains no C_k with $k \geq 5$.

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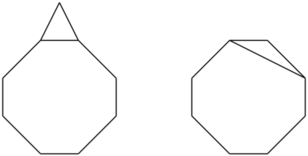

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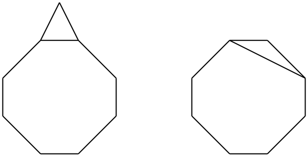

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Proof.

Mini plays so that "most" edges are in triangles.

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Proof.

Mini plays so that "most" edges are in triangles. This implies that the graph is C_k-free for all $k \geq 5$, and thus has at most $2n - 2$ edges at the end of the game. □