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To appear at the Constant Consonant Conference Concerning Combinatorics

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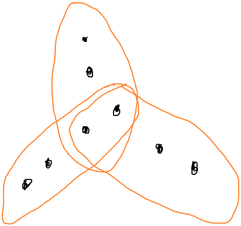
Recall that a *hypergraph* or *set system* $\mathcal{H}$ is a collection of sets called *edges*. The hypergraph is said to be *r-uniform* if every edge has size exactly r.



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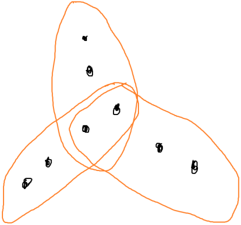

A hypergraph $\mathcal{S} = \{S_1, \ldots, S_p\}$ is called a *p-sunflower* if there exists a set K called the *kernel* such that $S_i \cap S_j = K$ for all $i \neq j$.

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Theorem (Erdős-Rado)

For all r, p, there exists a constant $f(r,p) = (pr)^r$ such that any r-uniform hypergraph $\mathcal{H}$ with more than $f(r,p)$ edges contains a p-sunflower.

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Theorem (Alweiss-Lovett-Wu-Zhang; Rao; Bell-Chueluecha-Warnke)

There exists a constant $C > 0$ such that for all r, p, any r-uniform hypergraph $\mathcal{H}$ with more than $(Cp \log r)^r$ edges contains a p-sunflower.

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Theorem (Frankston-Kahn-Narayanan-Park)

Let $\mathcal{H}$ be an r-uniform q-spread hypergraph with vertex set V. There exists an absolute constant C_0 such that if W is a uniformly random set of size $Cq \log r \cdot |V|$ chosen from V with $C \geq C_0$, then

$$\Pr[W \textit{ contains an edge of } \mathcal{H}] \geq 1 - \frac{C_0}{C \log r}.$$

I.e. a random set of proportion $q \log r$ is likely to contain an edge.

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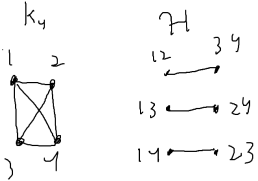
Let $\mathcal{H}$ be the hypergraph with vertex set $E(K_n)$ where each hyperedge S is a perfect matching of K_n.



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This result easily extends to perfect matchings in random r-uniform hypergraphs (which was previously thought to be much harder!)

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Proposition

Let F be an graph and define

$$t(F) = \max\left\{\frac{|E(F')|}{|V(F')|} : F' \subseteq F\right\}.$$

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It's not hard to prove this with a slightly fiddily second moment argument, but with spread hypergraphs the proof is much cleaner.

Applications

Let $\mathcal{H}$ be the hypergraph on $E(K_n)$ whose hyperedges correspond to copies of F. Note that each set $A \subseteq E(K_n)$ of positive degree in $\mathcal{H}$ corresponds to some subgraph $F_A \subseteq F$ with $|E(F_A)| = |A|$

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$$\left(\frac{d(A)}{|\mathcal{H}|}\right)^{1/|A|} \leq \left(\frac{n^{|V(F)|-|V(F_A)|}}{\binom{n}{|V(F)|}}\right)^{1/|A|} \approx n^{-|V(F_A)|/|E(F_A)|}.$$

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Thus $\mathcal{H}$ is q-spread with

$$q = \max\{n^{-|V(F')|/|E(F')|} : F' \subseteq F\} = n^{-1/t(F)}.$$

Plugging this into the theorem gives the result. □

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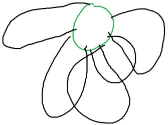

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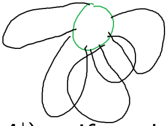

So by induction, the $(r - |A|)$-uniform hypergraph $\mathcal{H}' = \{S \setminus A : A \subseteq S \in \mathcal{H}\}$ contains a sunflower $\{S_1, \ldots, S_p\}$, which means $\mathcal{H}$ contains the sunflower $\{S_1 \cup A, \ldots, S_p \cup A\}$.

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Lemma

If $\mathcal{H}$ is a q-spread r-uniform hypergraph and you randomly choose a set $W \subseteq V(\mathcal{H})$ of size $q|V|$, then it's very likely that almost every $S \in \mathcal{H}$ has at least half its vertices covered, i.e. $|S \setminus W| \leq r/2$.

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Let's just pretend this is true for a second.

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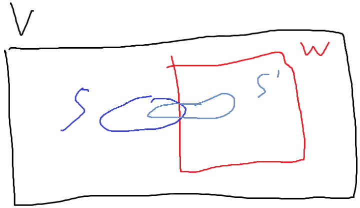
- After about $\log r$ steps, $\mathcal{H}_i$ is going to have some empty edges, i.e. there exists $S \in \mathcal{H}$ such that $S \subseteq W_1 \cup W_2 \cdots \cup W_i$.
- The set $W = W_1 \cup W_2 \cdots \cup W_i$ is basically a random set of size $q \log r|V|$, so we conclude that a set of this size is likely to contain an edge of $\mathcal{H}$. $\square$

Proof of Main Theorem

Given an r-uniform hypergraph $\mathcal{H}$, say that a pair of sets (S, W) with $S \in \mathcal{H}$ is *good* if there exists some edge $S' \subseteq S \cup W$ with $|S' \setminus W| \leq r/2$ and that it's *bad* otherwise



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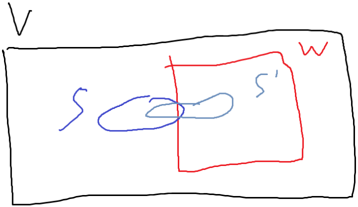

It turns out that the same approach as before works as long as almost all pairs (S, W) are good (e.g. we let $\mathcal{H}_2$ have edge set $S' \setminus W_1$ as opposed to $S \setminus W_1$).

Proof of Main Theorem

Lemma (Not False)

Let $\mathcal{H}$ be an r-uniform n-vertex hypergraph on V which is q-spread. If $p = Cq$, then

$$\left|\left\{(S, W) : S \in \mathcal{H},\ W \in \binom{V}{pn},\ (S, W) \textit{ is bad}\right\}\right| \leq 3(C/2)^{-r/4}|\mathcal{H}|\binom{n}{pn}$$

I.e. for large C almost every pair (S, W) is such that there exists $S' \subseteq S \cup W$ with $|S' \setminus W| \leq r/2$.

Proof of Main Theorem

Lemma (Not False)

Let $\mathcal{H}$ be an r-uniform n-vertex hypergraph on V which is q-spread. If $p = Cq$, then

$$\left|\left\{(S,W) : S \in \mathcal{H},\ W \in \binom{V}{pn},\ (S,W) \textit{ is bad}\right\}\right| \leq 3(C/2)^{-r/4}|\mathcal{H}|\binom{n}{pn}$$

I.e. for large C almost every pair (S, W) is such that there exists $S' \subseteq S \cup W$ with $|S' \setminus W| \leq r/2$. For $t \leq r$, define

$$\mathcal{B}_t = \{(S,W) : S \in \mathcal{H},\ W \in \binom{V}{pn},\ (S,W) \text{ is bad},\ |S \cap W| = t\}.$$

Observe that the quantity we wish to bound is $\sum_t |\mathcal{B}_t|$, so it suffices to bound each term of this sum.

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With this in mind, we (somewhat imprecisely) say that a bad pair (S, W) is pathological if the number of bad pairs in $S \cup W$ is larger than some quantity N to be determined later.

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Let $w := pn - t$.

Claim: The number of $(S, W) \in \mathcal{B}_t$ which are non-pathological is at most

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Claim: The number of $(S, W) \in \mathcal{B}_t$ which are pathological is at most

$$|\mathcal{H}| \cdot \binom{r}{t} \cdot 2(C/2)^{-r/2} |\mathcal{H}| \frac{\binom{w+r}{r}}{\binom{n}{r} N} \binom{n-r}{w}.$$

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$$\Pr[\#\{S' \subseteq S \cup W : |S' \cap S| \geq r/2\} \geq N] \leq \frac{\mathbb{E}[\#\{S' \subseteq S \cup W : |S' \cap S| \geq r/2\}]}{N}$$

(if (S, W) is bad then every $S' \subseteq S \cup W$ satisfies $|S' \cap S| \geq r/2$).

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We can now pick N so that the estimates of these two claims are about the same, and in total this shows there are few bad pairs, proving the lemma (and hence the theorem). □

Further Results

Theorem (Frankston-Kahn-Naryanan-Park)

If $\mathcal{H}$ is q-spread and r-uniform, then a random set of size $\gg q \log r \cdot |V|$ contains an edge of $\mathcal{H}$ with high probability.

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The proof is remarkably similar to the proof we just outlined, so it's perhaps natural to ask if we can (1) generalize when we can drop the $\log r$ term, and (2) try and find some interpolation between these two proof methods.

Further Results

We say that a hypergraph $\mathcal{H}$ is $(q; r_1, r_2, \ldots, r_\ell)$-spread with $r_1 > r_2 > \cdots > r_\ell$ if it's r_1-uniform and for every $A \subseteq V$ with $|A| = r_i$ and $j \geq r_{i+1}$

$$\sum_{B \subseteq A : |B| = j} d(B) \leq q^j |\mathcal{H}|.$$

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Proposition

We have the following.

(a) *If $\mathcal{H}$ is $(q; r_1, \ldots, r_\ell, 1)$-spread, then it is q-spread.*

(b) *If $\mathcal{H}$ is q-spread and r-uniform, then it is $(4q; r, r/2, \ldots, 1)$-spread.*

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Theorem (S.)

If $\mathcal{H}$ is $(q; r_1, r_2, \ldots, r_\ell, 1)$-spread, then a random set of size $q\ell|V|$ is very likely to contain an edge of $\mathcal{H}$.

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The proof is basically the same as before, except now instead of going from edges of size $r, r/2, r/4, \ldots$ we do $r_1, r_2, r_3, \ldots$ (and the definition is designed precisely so that the proof still works).

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The proof is basically the same as before, except now instead of going from edges of size $r, r/2, r/4, \ldots$ we do $r_1, r_2, r_3, \ldots$ (and the definition is designed precisely so that the proof still works). This theorem succeeds in recovering/interpolating between basically all previously known results.

The End

Thank You!