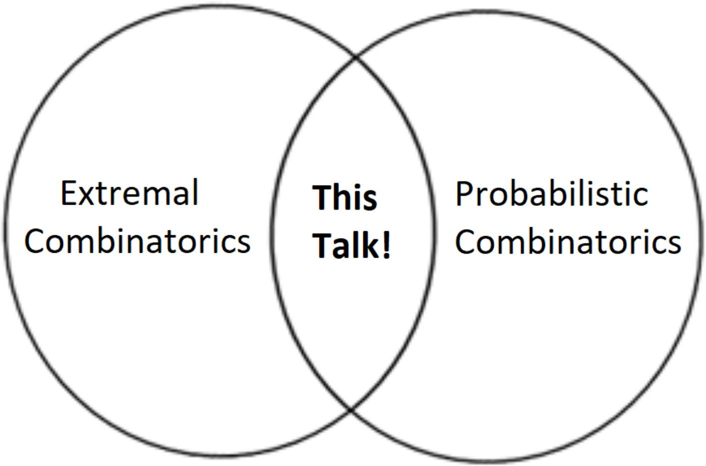
Extremal Problems for Random Objects

Sam Spiro, Rutgers University


Extremal
Combinatorics
This
Talk!
Probabilistic
Combinatorics

Part I: Card Guessing with Feedback

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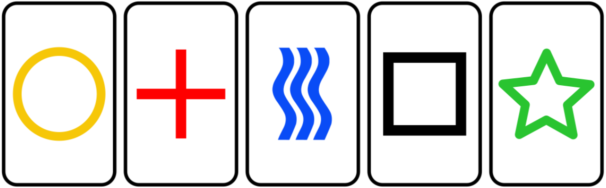

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Theorem (Diaconis-Graham, 1981)

For n fixed,

$$\mathcal{C}^{\pm}_{m,n} = m \pm c_n\sqrt{m} + o_n(\sqrt{m}).$$

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What happens when n is large?

Theorem (Diaconis-Graham-He-S., 2020)

For m fixed,

$$\mathcal{C}^{+}_{m,n} \sim H_m \log(n),$$
$$\mathcal{C}^{-}_{m,n} = \Theta(n^{-1/m}),$$

where H_m is the mth harmonic number.

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With this we have the trivial bounds

$$m \leq \mathcal{P}^{+}_{m,n} \leq \mathcal{C}^{+}_{m,n} = O_m(\log n).$$

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Theorem (Diaconis-Graham-He-S., 2020)

There exist $c, C > 0$ such that if n is sufficiently large in terms of m, we have

$$m + c\sqrt{m} \leq \mathcal{P}^{+}_{m,n}$$

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There exist $c, C > 0$ such that if n is sufficiently large in terms of m, we have

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Theorem (Z. Nie, 2022)

If $n \gg m$, then

$$\mathcal{P}^+_{m,n} = m + \Theta(\sqrt{m}).$$

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$$\Pr[\pi_t = i] \leq \frac{m}{mn - g_i - S}.$$

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More precisely, we are now considering a two player game played by Shuffler and Guesser. Let $\mathcal{C}_{m,n}(G, S)$ be the expected number of points Guesser scores when the two players follow strategies G, S.

$$\Theta_m(n^{-1/m}) \leq \mathcal{C}_{m,n}(G, \mathrm{Uniform}) \leq H_m \log n + o_m(\log n).$$

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Theorem (S., 2021)

There exists a strategy S' *for Shuffler so that*

$$\mathcal{C}_{m,n}(\mathsf{G}, \mathsf{S}') \leq \log n + o_m(\log n),$$

and this bound is best possible.

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Interestingly, the greedy strategy is also the “unique” strategy which maximizes the number of correct guesses if Guesser tries to minimize their score.

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Prove non-trivial bounds for the partial feedback model with adversarial shufflings.

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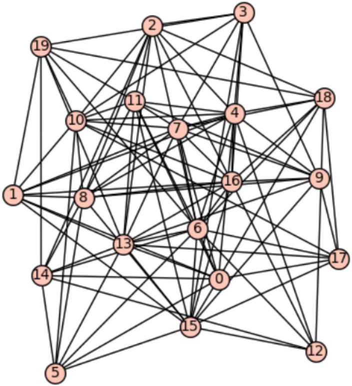
Conjecture

The minimum expected score one can get with partial feedback is asymptotic to m.

Part II: Turán's Problem in Random Graphs



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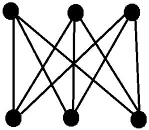
Theorem (Mantel 1907)

$$\mathrm{ex}(n, K_3) = \left\lfloor n^2/4 \right\rfloor .$$



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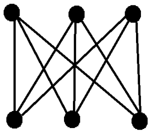

Theorem (Erdős-Stone 1946)

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}.$$

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The lower bound is tight when $p = 1$. The upper bound is tight if p is "small."

$$\frac{1}{2}p\binom{n}{2} \lesssim \mathrm{ex}(G_{n,p}, K_3) \lesssim p\binom{n}{2},$$

with the lower bound tight for $p = 1$ and the upper bound tight for $p \ll n^{-1/2}$.

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Theorem (Conlon-Gowers, Schacht 2010)

Whp,

$$\mathrm{ex}(G_{n,p}, F) = p \cdot \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)\binom{n}{2} \qquad p \gg n^{-1/m_2(F)},$$

where $m_2(F) = \max\{\frac{e(F')-1}{v(F')-2} : F' \subseteq F\}$.

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Conjecture

If F is a bipartite graph which is not a forest, then whp

$$\mathrm{ex}(G_{n,p}, F) = \begin{cases} \Theta(p \cdot \mathrm{ex}(n, F)) & p \gg n^{-1/m_2(F)}, \\ (1 + o(1))p\binom{n}{2} & p \ll n^{-1/m_2(F)}. \end{cases}$$

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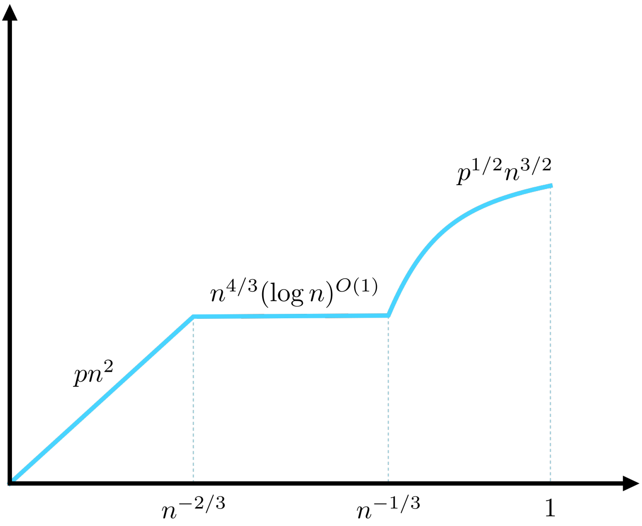
This conjecture turns out to be completely false!



Plot of $\mathsf{ex}(G_{n,p}, C_4)$ (Füredi 1991)

Conjecture (McKinley-S.)

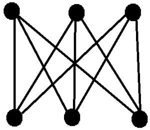
If F is a graph with $\mathrm{ex}(n, F) = \Theta(n^\alpha)$ for some $\alpha \in (1, 2]$, then whp

$$\mathrm{ex}(G_{n,p}, F) = \max\{\Theta(p^{\alpha-1} n^\alpha), n^{2-1/m_2(F)} (\log n)^{O(1)}\},$$

provided $p \gg n^{-1/m_2(F)}$.

Theorem (Kővari-Sós-Turán 1954)

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s}).$$



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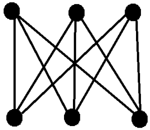

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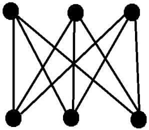

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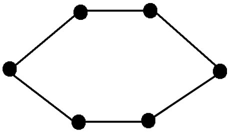
Moreover, this bound is tight whenever $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$.

Theorem (Bondy-Simonovits 1974)

$$\mathrm{ex}(n, C_{2b}) = O(n^{1+1/b}).$$



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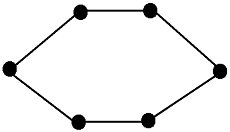

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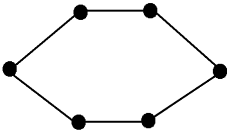

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Moreover, this is tight whenever $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2b}\}) = \Theta(n^{1+1/b})$.

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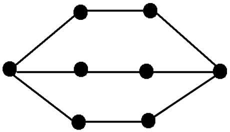
$$\mathrm{ex}(n, \theta_{a,b}) = O(n^{1+1/b}).$$



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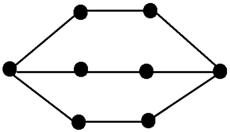

Theorem (McKinley-S. 2023)

For $a \geq 100$,

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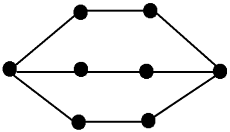

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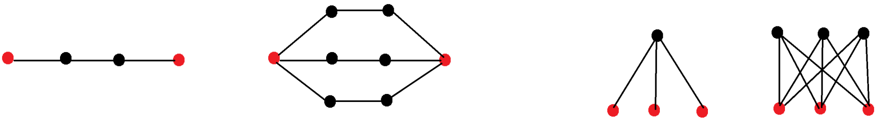
Moreover, this bound is tight whenever a is sufficiently large in terms of b.

Theorem (Bukh-Conlon 2015)

If T^ℓ is the "ℓth power of a balanced tree" and ℓ is sufficiently large, then

$$\mathrm{ex}(n, T^\ell) = \Omega(n^{2-\rho(T)}).$$



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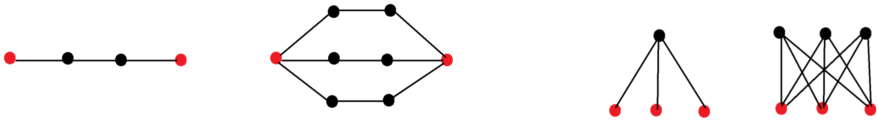
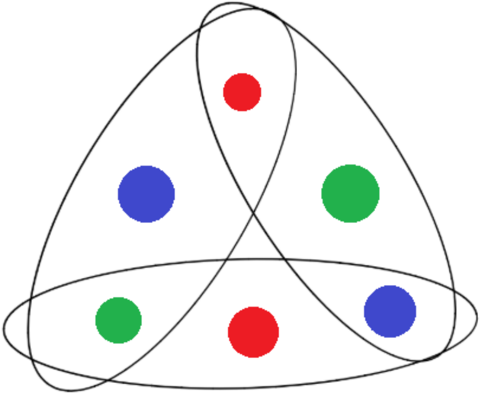


Theorem (S. 2022)

$$\mathrm{ex}(G_{n,p}, T^\ell) = \Omega(p^{1-\rho(T)} n^{2-\rho(T)}),$$

provided ℓ is sufficiently large.

Hypergraphs



Theorem (S.-Verstraëte 2021)

Let $K^r_{s_1,\ldots,s_r}$ denote the complete r-partite r-graph with parts of sizes $s_1, \ldots, s_r$. There exist constants $\beta_1, \beta_2, \beta_3, \gamma$ depending on $s_1, \ldots, s_r$ such that, for s_r sufficiently large in terms of $s_1, \ldots, s_{r-1}$, we have whp

$$\mathrm{ex}(G^r_{n,p}, K^r_{s_1,\ldots,s_r}) = \begin{cases} \Theta\left(pn^r\right) & n^{-r} \ll p \leq n^{-\beta_1}, \\ n^{r-\beta_1+o(1)} & n^{-\beta_1} \leq p \leq n^{-\beta_2}(\log n)^{\gamma}, \\ \Theta(p^{1-\beta_3} n^{r-\beta_3}) & n^{-\beta_2}(\log n)^{\gamma} \leq p \leq 1. \end{cases}$$

Question

Does the McKinley-Spiro conjecture extend to hypergraphs?

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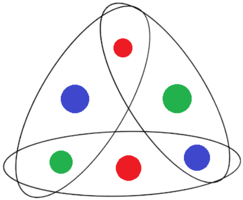
Theorem (Nie-S. 2023 (Informal))

Many hypergraphs fail to have a flat middle range.

Other Hypergraph Results

1. Solved for loose triangles (Nie-S.-Verstraëte 2020; Nie 2023)
2. Solved for loose even cycles of uniformity $r \geq 4$ (Mubayi-Yepremyan 2020; Nie 2023)
3. (Non-optimal) bounds for Berge cycles (S.-Verstraëte 2021; Nie 2023)
4. *Improved lower bound for non-Sidorenko hypergraphs (Nie-S. 2023)
5. *Lifting upper bounds from graphs to hypergraphs (Nie-S. 20XX++)

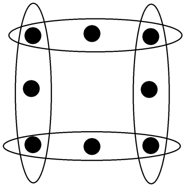


Future Problems

Future Problems

Problem

Prove tight bounds for the 3-uniform loose 4-cycle.



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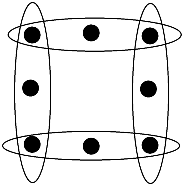

Problem

Prove tight bounds for subdivisions of complete bipartite graphs.

Thanks!