Triangle-Free Subgraphs of Hypergraphs

Jiaxi Nie, Sam Spiro*, Jacques Verstraete

Introduction

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What about triangle-free *hypergraphs*?

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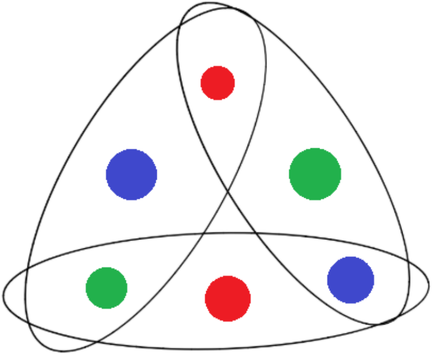

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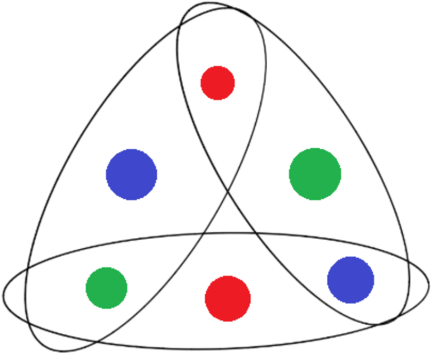

Observe that for $r \geq 3$, T^r is r-partite.

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Theorem (Frankl-Füredi, 1987)

For $r \geq 3$ and n sufficiently large,

$$\mathrm{ex}(n, T^r) = \binom{n-1}{r-1},$$

with the extremal example being the star $S_{n,r}$ which has all r-sets containing 1.

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so in terms of order of magnitude the main case of interest is when F is r-partite.

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To find a large triangle-free subgraph of H, we will use a triangle-free 3-graph J as a "template."

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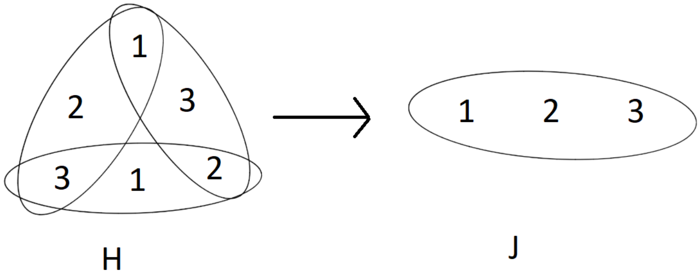

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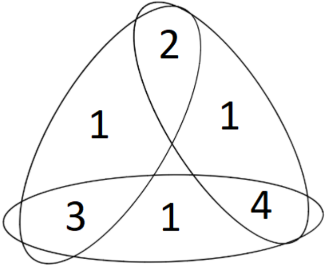

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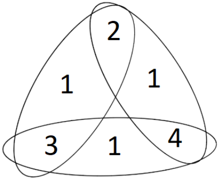

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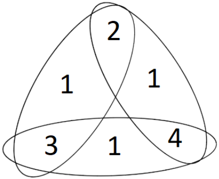

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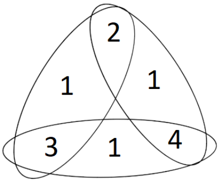

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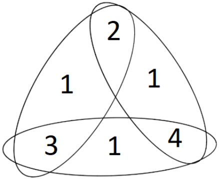

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For all t there exists a 3-graph R_t on t vertices which is triangle-free with $t^{2-o(1)}$ edges

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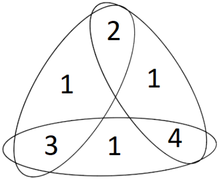

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Linearity of expectation then gives $\mathbb{E}[e(H')] \geq \Delta^{-1/2-o(1)} e(H)$. □

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There exists an r-graph H with

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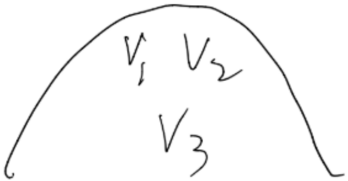

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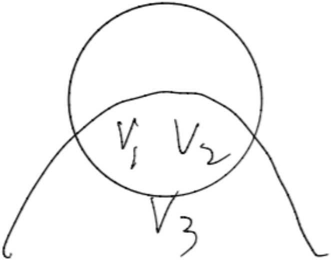

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There exist a set $E_2 \subseteq E(H')$ of $2r+1$ edges containing v_1, v_3.
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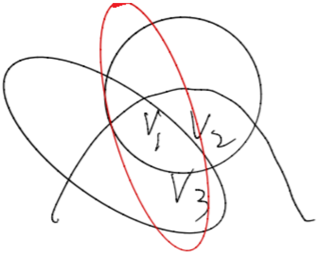

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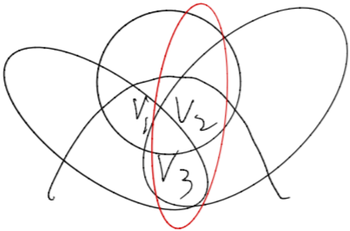

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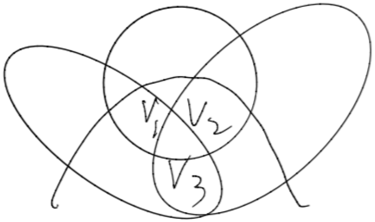
Thus H' had a T^r, a contradiction, so H'' must be empty. □



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If $pn^3 \to \infty$, then a.a.s.

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Note that $p^{1/3}n^2 \geq \Delta^{-1/2}e(G^3_{n,p})$, so we get a stronger result in this range.

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The key change is that this holds for $t \ll \Delta^{1/2}$ since we don't multiply this by $1 - 27\Delta t^{-2}$.

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Let $H'' \subseteq H'$ be obtained by deleting an edge from each T^3 in H'. We have that H'' is triangle-free and

$$\mathbb{E}[e(H'')] \geq e(H)t^{-1-o(1)} - R(H)t^{-4-o(1)}.$$

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For $H = G^3_{n,p}$ we have with high probability that $e(H) \approx pn^3$ and $R(H) \approx p^3n^6$. Taking $t \approx p^{2/3}n$ (which is at least 1 for $p \gg n^{-3/2}$) gives a lower bound of $p^{1/3}n^{2-o(1)}$ as desired. □

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Random Hosts

Lemma

For any integer n and positive number t with $12 \leq t \leq \binom{n}{3}/n^2$, there exists a collection $\mathcal{C}$ of subgraphs of K_n^3 such that for some constant c:

(1) *For any T^3-free subgraph J of K_n^3, there exists $C \in \mathcal{C}$ such that $J \subset C$.*

(2) $|\mathcal{C}| \leq \exp\left(\frac{c \log(t) n^2}{\sqrt{t}}\right)$.

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The key ingredients is the standard container lemma of Balogh, Morris and Samotij; together with a supersaturation result for triangles due to Balogh, Narayanan, and Skokan.

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To show that $\mathrm{ex}(G^3_{n,p}, T^3) < m := p^{1/3}n^2$, define X_m to be the number of triangle-free subgraphs of $G^3_{n,p}$ on m edges. We have $\mathbb{E}[X_m] = N_3(n,m)p^m \to 0$, so we conclude the result by Markov's inequality. □

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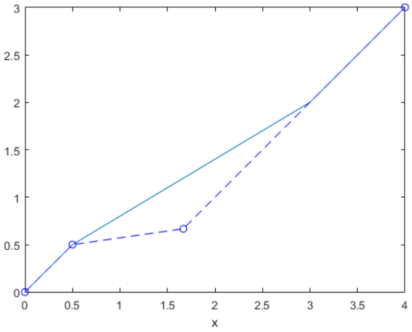

Bounds for $f_5(x)$.

Generalizing to other F

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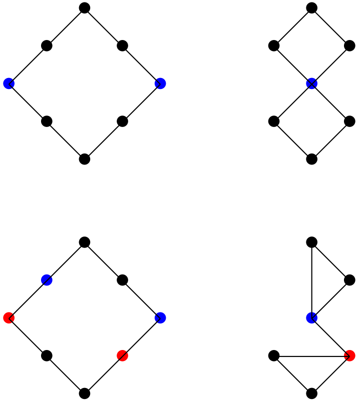
For (2), we say that a map $\chi : V(F) \to V(F')$ is a local isomorphism if (a) it is a homomorphism with induced map $\chi^* : E(F) \to E(F')$ and (b) if $|e \cap f| \neq 0$, then $\chi^*(e) \neq \chi^*(f)$.

Generalizing to other F

For example, we have local isomorphisms from C_8 to the following graphs



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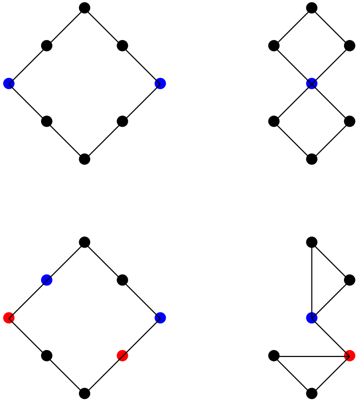

We define $\mathcal{H}_F$ to be the set of F' such that there exists a surjective local isomorphism from F to F'.

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If Δ_2 isn't much larger than $\Delta^{1/2}$, then this gives pretty good bounds. If Δ_2 is very large, we have to do a different kind of construction, which roughly has us locating a "matching" of 2-sets which have large codegree. While this second method works for some F, in general we don't know how to get a good construction when Δ_2 is large.

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Theorem (Nie-S.-Verstraete, 2020+)

If s is sufficiently large and H has maximum degree Δ, then

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so the construction here is *not* just a clique.

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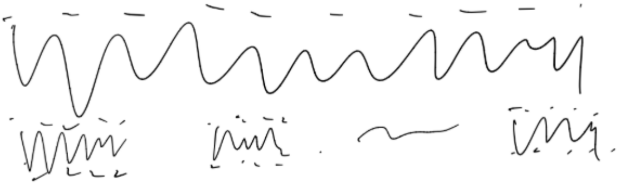
The construction: define $H(n)$ to be the 3-graph on vertex set $\{x_i : 1 \leq i \leq n^2\} \cup \{y_{i,j}, z_{i,j} : 1 \leq i, j \leq n\}$ with all edges of the form $\{x_i, y_{i',j}, z_{i'',j}\}$.



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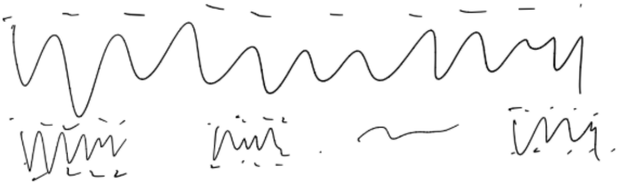

This turns out to give the desired bounds.

The End

Thank You!