

Solutions for Advanced Graph Theory

Sam Spiro*

February 5, 2026

Due to limited time the following are only **sketches** of full solutions, and in particular these solutions alone wouldn't necessarily constitute a solution worth full marks. I also emphasize that there may exist other (and possibly simpler) solutions these problems.

1 HW1

0.1 (*Handshaking Lemma*) Prove that every graph G has $\sum_{x \in V(G)} \deg(x) = 2e(G)$ [2-].

Consider the set of pairs $\mathcal{P} = \{(v, e) : v \in V(G), e \in E(G), v \in e\}$ and for each vertex v let $\mathcal{P}_v = \{(v, e) : e \in E(G), v \in e\}$. Then the \mathcal{P}_v sets partition \mathcal{P} , giving

$$2e(G) = |\mathcal{P}| = \sum |\mathcal{P}_v| = \sum \deg(v).$$

Alternatively you could consider the set of pairs $\{(v, w) : vw \in E(G)\}$ which gives a similar argument.

0.4 Prove that a graph is bipartite if and only if it contains no odd cycles [2-].

If G has an odd cycle $(v_1, \dots, v_{2\ell+1})$ and a bipartition $V_1 \cup V_2$ with, say, $v_1 \in V_1$, then one can prove inductively that $v_i \in V_j$ iff $i \equiv j \pmod{2}$. But this implies $v_1 v_{2\ell+1} \in E(G)$ has both vertices in V_1 , a contradiction.

For the other direction, one should prove (or at least cite) (i) a graph has an odd cycle if and only if it has a closed walk of odd length, and (ii) a graph is bipartite if and only if each of its connected components is bipartite. With this, if we assume our graph has no odd cycles then you can define a bipartition on each connected component by picking an arbitrary vertex v and defining $V_1 \cup V_2$ by having $u \in V_1$ iff $\text{dist}(u, v)$ is odd. One can check that this can not have, say, $uu' \in E(G)$ with $u, u' \in V_1$ as otherwise this plus the paths from u, u' to v would create an odd closed walk.

1.3a Prove that

$$z(m, n; s, t) \leq (t-1)^{1/s} m n^{1-1/s} + (s-1)n.$$

(Hint: if you're struggling with this, try solving the previous problem first) [2].

*Dept. of Mathematics and Statistics, sspiro@gsu.edu

Consider the set \mathcal{P} of pairs (S, v) where $v \in V$ and $S \subseteq N(v)$ is a set of size s . If G avoids a $K_{s,t}$ then each of the $\binom{m}{s}$ sets S can belong to at most $t - 1$ pairs, proving $|\mathcal{P}| \leq (t - 1)\binom{m}{s}$. On the other hand a convexity argument shows $|\mathcal{P}| \geq n \binom{n^{-1}e(G)}{s}$, and now some algebra gives the result.

- 1.3b *Prove that if G is an n -vertex bipartite C_4 -free graph then $e(G) \leq 2^{-3/2}n^{3/2} + o(n^{3/2})$, i.e. the lower bound we got for $\text{ex}(n, C_4)$ using G_q was best possible in the setting of bipartite graphs [2].*

Let G be such a graph with parts of sizes m and $n - m$, and (importantly) without loss of generality we may assume $m \leq n/2$. By the first part with $s = t = 2$ we have $e(G) \leq m(n - m)^{1/2} + (n - m) = m(n - m)^{1/2} + o(n^{3/2})$. One can check (using calculus, for example), that this expression is maximized in the range $0 \leq m \leq n/2$ at the value $m = n/2$, giving the bound. Note crucially that if we did not assume $m \leq n/2$ then the maximum would be at $m = 2n/3$, which would give a suboptimal bound.

- 1.3c *Prove that for all s, t there exists a constant $C > 0$ such that if G is an n -vertex $K_{s,t}$ -free graph, then the number of edges $xy \in E(G)$ with $\deg(x) \geq Cn^{1-1/s}$ is at most $O(n)$. Find an example of a graph which has $\Theta(n)$ edges of this form (Hint: the intended proof I have in mind works with $C \approx (s + t - 1)^{1/s}$) [2].*

Define an auxilliary bipartite graph B where one part U consists of all vertices of G with $\deg_G(x) \geq Cn^{1-1/s}$ and the other part V consists of a disjoint copy of $V(G)$ where we have $xy \in E(B)$ if $xy \in E(G)$ and $x \in U$. Observe that B can not contain a $K_{s,t+s}$ with the part of size s in U , since if it did then removing the at most s vertices that appear in both parts of the $K_{s,t+s}$ from the part of size $t + s$ would give a $K_{s,t}$ in G . It follows from (a) that $e(B) \leq (s + t - 1)^{1/s}|U|n^{1-1/s} + (s - 1)n$. On the other hand, by construction we have $e(B) \geq Cn^{1-1/s}|U|$, so if say $C = 2(s + t - 1)^{1/s}|U|n^{1-1/s}$ then this implies $(s - 1)n \geq (s + t - 1)^{1/s}|U|n^{1-1/s}$, and hence $e(B) \leq (s + t - 1)^{1/s}|U|n^{1-1/s} + (s - 1)n \leq 2(s - 1)n$. But $e(B)$ is exactly the number of edges of the form that we wish to bound, proving the result.

An example of a graph which this is tight is an n -vertex star.

- 1.4a *Prove that if G is an n -vertex graph then G contains at least $e(G) - \text{ex}(n, F)$ copies of F for any graph F with at least one edge [1].*

One can prove this by induction on $\Delta := e(G) - \text{ex}(n, F)$, for example, the case $\Delta = 0$ being trivial. Inductively for $\Delta > 0$, we have by definition of $\text{ex}(n, F)$ that G contains some copy of F , let e be any such edge. Then inductively $G - e$ contains at least $\Delta - 1$ copies of F and these copies must be distinct from the copy of F containing e (since these copies live in $G - e$), giving the desired result.

- 1.4b *Prove that if G is an n -vertex graph with $e(G) \geq 100n^{3/2}$ then G contains at least $\Omega(n^{-4}e(G)^4)$ copies of C_4 (the number 100 does not matter in case you'd rather prove this result with a different constant) [2].*

For notational convenience let $d(u, v) = |N(u) \cap N(v)|$. It is not too difficult to see that

the total number of C_4 's in a graph G is $\sum_{u \neq v} \binom{d(u,v)}{2}$. By convexity this is at least

$$\binom{n}{2} \left(\binom{n}{2}^{-1} \sum \frac{d(u,v)}{2} \right) \approx n^{-2} (\sum d(u,v))^2.$$

Note that $\sum d(u,v) = |\mathcal{P}|$ the set of pairs that we defined in class, and this in turn is lower bounded by roughly $n^{-1}e(G)^2$ by the convexity argument we did in class, and plugging this in gives the desired result.

2 HW2

- 1.8 *Verify that if G' is an n -vertex complete $(r-1)$ -partite graph then $e(G') \leq e(T_{r-1}(n))$ [1+].*

Let n_1, \dots, n_{r-1} be the sizes of a complete $(r-1)$ -partite graph on n vertices. Then its number of edges equals

$$\sum_{i < j} n_i n_j.$$

If there exists some i, j with say $n_i \geq 2 + n_j$, then one can consider a new sequence of integers defined by replacing n_i, n_j with $n_i - 1, n_j + 1$ and one can easily check that this strictly increases the sum above. As such, the sum is maximized when all of these integers are within 1 of each other, and this in turn is only possible if each value is equal to the floor or ceiling of $n/(r-1)$ (since in particular, some value must be at least the floor and some value must be at least the ceiling simply by averaging).

- 1.9a *Observe that if G is a triangle-free graph, then $\deg(x) + \deg(y) \leq v(G)$ for all $xy \in E(G)$. Use this to prove Mantel's Theorem (which is in fact the original way Mantel proved his result) [2].*

Let G be an n -vertex triangle-free graph. Then our observation above implies

$$\sum_{xy \in E(G)} \deg(x) + \deg(y) \leq ne(G).$$

On the other hand, each term $\deg(x)$ in this sum appears exactly $\deg(x)$ times, meaning the sum equals $\sum_x \deg(x)^2$. Note that Cauchy-Schwarz implies that for any sequence of n real numbers x_i that $n \sum_i x_i^2 = \sum_i 1^2 \cdot \sum_i x_i^2 \geq (\sum_i x_i)^2$. Applying this here gives $\sum \deg(x)^2 \geq n(2e(G))^2$ which exactly gives the bound that we want.

- 1.9b *Generalize our inductive proof of Mantel's Theorem to give an alternative proof of Turán's Theorem (which is in fact the original way that Turán proved his result).*

Instead of deleting a pair of vertices in an edge like we did for Mantel, we now delete a set of vertices forming a K_{r-1} . In this case no vertex outside the K_{r-1} can be adjacent to every vertex of this K_{r-1} , and the same sort of analysis as we did before proves the result.

- 1.10 Let F denote the unique 4-vertex graph with 5 edges (i.e. the graph consisting of two triangles sharing an edge). Prove (without using ??) that $\text{ex}(n, F) = \lfloor n^2/4 \rfloor$ for all $n \geq 4$.

We prove this by induction on n , the base cases being straightforward. If we have a graph with more than $n^2/4$ edges then we know there exists some triangle xyz . Look at $G - x - y - z$ and observe now that every vertex here has at most one neighbor in x, y, z since otherwise we create our forbidden graph. Then the same analysis we did for our previous inductive proof gives the result.