Intro to Topology

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1 What is Topology?

This question is somewhat complicated to answer because people use the word "topology" to refer to two related but distinct areas of study:

- *Modern topology* is roughly speaking the study of geometric objects such as spheres, Möbius strips, Klein bottles, and so on. If a mathematician says they "study topology", this is typically what they're referring to.

- *Point set topology* (also called *general topology*) is a very general framework that includes modern topology, much of calculus, and many other areas of mathematics. **This** is what the present course is all about, and from now on whenever I say the word "topology" I'll be referring to this concept.

The central object studied in topology are mathematical objects called *topologies*. So again we're left with the question: what is a topology?

1.1 What is a Topology: the Short Answer

The definition for a topology is a follows. At this point it should **not** be obvious to you why in the world you would ever consider something like this. Here and throughout, given a set X we let $\mathcal{P}(X)$ denote the power set of X, i.e. the set of all subsets of X.

Definition 1. Given a set X, a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* of X if the following conditions are satisfied:

(a) $\emptyset, X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under (arbitrary) unions, i.e. for any $\mathcal{S} \subseteq \mathcal{T}$, we have $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

(c) $\mathcal{T}$ is closed under *finite* intersections, i.e. for any *finite* subset $\mathcal{S} \subseteq \mathcal{T}$, $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

Again, this is a strange definition that should seem totally bizarre. In the next subsection I'll attempt to provide motivation for why one might possibly come up with this definition. Those that aren't interested/confused by this discussion can completely ignore it without affecting their understanding of the rest of the material in this course.

1.2 What is Topology: the Long Answer

The study of topology originates with the study of calculus/real analysis. When you took courses in these areas, you learned a number of important concepts about the set of real numbers $\mathbb{R}$, as well as about functions f from $\mathbb{R}$ to $\mathbb{R}$. In particular, two very important definitions are:

1. What it means for a sequence of real numbers $(x_n)_{n\geq 1}$ to *converge* to a real number x_0.

2. What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous*.

The central aim of topology is to give a *general framework* which expands these definitions for real numbers to a much broader class of mathematical objects. In particular, it aims to answer the following two questions:

1. What does it mean for a sequence of "objects" $(x_n)_{n\geq 1}$ to *converge* to another object x_0? For example, what does it mean for a sequence of functions $(f_n)_{n\geq 1}$ to converge to another function?

2. What does it mean for a function $f : X \to Y$ between two "nice objects" X, Y to be *continuous*? For example, what does it mean for a map $f : S^2 \to S^2$ from the sphere to itself to be continuous?

We'll postpone the second question and focus on convergence. Of course, any "reasonable" answer should in particular recover the original definition of convergence from real analysis. With this in mind, let's recall this definition and then think about how we might generalize it.

Definition 2. We say that a sequence of real numbers $(x_n)_{n\geq 1}$ *converges* to a real number x_0 if for all $\varepsilon > 0$, there exists an integer N_ε such that $|x_n - x_0| < \varepsilon$ for all $n \geq N_\varepsilon$.

While this is a fine definition, it's a little difficult to generalize. It turns out (for reasons that should not be obvious at this point) that a "better" definition can be made by utilizing the language of *open intervals*, which we recall are sets $I \subseteq \mathbb{R}$ of the form $\{x : a < x < b\}$ for some $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 3. We say that a sequence of real numbers $(x_n)_{n\geq 1}$ *converges* to a real number x_0 if for every open interval I containing x_0, there exists an integer N_I such that $x_n \in I$ for all $n \geq N_I$.

Claim 1.1. *These two definitions are equivalent. That is, a sequence $(x_n)_{n\geq 1}$ and real number x_0 satisfy the conditions of Definition 2 if and only if they satisfy the conditions of Definition 3.*

Definition 3 has several advantages over Definition 2. First, it avoids any mention of the real number ε (which is mathematically nice[1] since we ultimately want to generalize things beyond real numbers). Second, it frames the definition in terms of the more "geometric" concept of open intervals.

Now, at this point it probably still isn't obvious how to generalize Definition 3 to more general objects (like sequences of functions). However, if you were to spend 30 years about this problem, then perhaps you would come up with the following idea: replace the words "open interval" in Definition 3 with the words "nice set" (where the exact definition of "nice set" depends on your exact problem at hand) and use this as your definition of convergence. Somewhat more precisely, we'll try to work with the following definition (which at this point in time you don't need to memorize since we'll forget about it a moment).

Definition 4. Given a set X, we call any set $\mathcal{T} \subseteq \mathcal{P}(X)$ a *pre-topology*[2] of X. We say that a sequence of points $(x_n)_{n\geq 1}$ with $x_n \in X$ *converges* to a point $x_0 \in X$ with respect to $\mathcal{T}$ if for every $I \in \mathcal{T}$ containing x_0, there exists an integer N_I such that $x_n \in I$ for all $n \geq N_I$.

For example, if $X = \mathbb{R}$ and $\mathcal{T}$ is the set of open intervals of $\mathbb{R}$, then this exactly recovers Definition 3. Here are a few more (extreme) examples to give some more familiarity with these definitions.

Claim 1.2. *Let X be an arbitrary set.*

*(a) If $\mathcal{T} = \emptyset$ (i.e. if $\mathcal{T}$ contains no subsets of X), then **every** sequence of points $(x_n)_{n\geq 1}$ in X converges to **every** point $x_0 \in X$ with respect to $\mathcal{T}$*

(b) If $\mathcal{T} = \mathcal{P}(X)$ (i.e. if $\mathcal{T}$ contains every subset of X), then a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point $x_0 \in X$ with respect to $\mathcal{T}$ if and only if there exists some N such that $x_n = x_0$ for all $n \geq N$ (i.e. iff x_n is "eventually constant").

If you play around with these definitions some more, you'll quickly find out that you can have two different pre-topologies $\mathcal{T}, \mathcal{T}'$ which are "equivalent" to each other in the following sense.

Definition 5. Two pre-topologies $\mathcal{T}, \mathcal{T}'$ for the same set X are said to be *equivalent* if: a sequence $(x_n)_{n\geq 1}$ converges to x_0 with respect to $\mathcal{T}$ if and only if it converges to x_0 with respect to $\mathcal{T}'$.

Again if you think about this concept for long enough you might end up asking the following.

Question 1.3. *Given a pre-topology $\mathcal{T}$, what is the "largest" pre-topology $\mathcal{T}'$ which contains $\mathcal{T}$ and which is equivalent to $\mathcal{T}$?*

The idea here is that while the pre-topologies $\mathcal{T}, \mathcal{T}'$ might be equivalent when it comes to convergence, the extra elements in $\mathcal{T}'$ give you extra flexibility in your proofs, and hence for certain applications would be more convenient to work with. This question seems a little daunting, so instead we ask the following weaker question.

[1]This is also psychologically nice for those who have painful memories of ε from real analysis.

[2]This name is not standard at all and we will never use it beyond this first pre-lecture.

Question 1.4. *Given a pre-topology $\mathcal{T}$, are there any "obvious" sets U that we can add to $\mathcal{T}$ so that $\mathcal{T} \cup \{U\}$ is equivalent to $\mathcal{T}$?*

Again if you think about this for 30 years you might realize the following.

Claim 1.5. *Let X be a set, $\mathcal{T}$ a pre-topology, and $\mathcal{S} \subseteq \mathcal{T}$ some non-empty subset of its elements.*

(a) $\mathcal{T} \cup \{\emptyset\}$ is equivalent to $\mathcal{T}$.

(b) $\mathcal{T} \cup \{X\}$ is equivalent to $\mathcal{T}$.

(b) $\mathcal{T} \cup \{\bigcup_{U \in \mathcal{S}} U\}$ is equivalent to $\mathcal{T}$.

(c) If $\mathcal{S}$ is a finite set, then $\mathcal{T} \cup \{\bigcap_{U \in \mathcal{S}} U\}$ is equivalent to $\mathcal{T}$.

(d) Part (c) does not hold in general if $\mathcal{S}$ is allowed to be an infinite set[3].

Sketch of Proof. For (a), since $\emptyset$ contains no elements of X it doesn't affect whether any given element is the limit of a sequence.

For (b), one can always take $N_X = 1$ (since every sequence lies in X for all $n \geq 1$).

For (c), take $N_{\bigcup_{U \in \mathcal{S}} U}$ equal to N_U for any $U \in \mathcal{S}$.

For (d), take $N_{\bigcap_{U \in \mathcal{S}} U} = \max_U N_U$ (note how this requires the set $\mathcal{S}$ to be finite). □

That is, given any pre-topology $\mathcal{T}$, we can freely add in $\emptyset$ and X, as well as (arbitrary) unions and finite intersections of elements of $\mathcal{T}$ into $\mathcal{T}$ to make a (possibly) larger pre-topology which is equivalent to $\mathcal{T}$. In particular, this means that the largest pre-topology $\mathcal{T}'$ which is equivalent to $\mathcal{T}$ must contain $\emptyset, X$ and be "closed" under taking unions and finite intersections. This is exactly the definition of a topology!

2 Definitions and Examples

Let's again restate the definition of a topology, as well as some related definitions that will serve as a useful language for talking about topologies.

Definition 6. Given a set X, a set $\mathcal{T}$ of subsets of X is called a *topology* of X if the following hold:

(a) $\emptyset, X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under arbitrary unions. That is, for any subset $\mathcal{S} \subseteq \mathcal{T}$, the set $\bigcup_{U \in \mathcal{S}} U$ is in $\mathcal{T}$.

[3] Hint: take $X = \mathbb{R}$, $\mathcal{T}$ to be the set of open intervals, and $\mathcal{S} = \{(-\frac{1}{n}, \frac{1}{n})\}$.

(c) $\mathcal{T}$ is closed under finite intersections. That is, for any finite subset $\mathcal{S} \subseteq \mathcal{T}$, the set $\bigcap_{U \in \mathcal{S}} U$ is in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*. We will call the pair $(X, \mathcal{T})$ a *topological space*. When $\mathcal{T}$ is clear from context we simply write X instead of $(X, \mathcal{T})$.

Remark 2.1. For arbitrary unions, the book likes to use the notation $\bigcup_{\alpha \in J} U_\alpha$ where J is an "index set", and we will occasionally use this notation in class as well. I recommend looking at Chapter 1 Section 5 of the book to get a more detailed explanation for how this notation is used throughout the book.

Now that we have the definition of a topology in hand, let's pause for a moment and look at some examples and non-examples of topologies.

2.1 Finite Topologies

Is the following pair $(X, \mathcal{T})$ a topological space:

$$X = \{a, b, c\},\ \mathcal{T}_1 = \{\{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

No! It fails to have $\emptyset \in \mathcal{T}_1$. Okay, what about

$$X = \{a, b, c\},\ \mathcal{T}_2 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

Well, we have $\emptyset, X \in \mathcal{T}_2$, and it is not difficult to check by hand that this is closed under unions and (finite) intersections (e.g. $\{a, b\} \cap \{b\} = \{b\} \in \mathcal{T}_2$; the slicker way is to note that unions/intersections of sets containing b continue to be sets containing b) so this *is* a topology! What about

$$X = \{a, b, c\},\ \mathcal{T}_3 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}?$$

No! This isn't closed under intersections $\{a, b\} \cap \{a, c\} = \{a\} \notin \mathcal{T}$. What about

$$X = \{a, b, c\},\ \mathcal{T}_4 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

No! Again not closed under intersections $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_4$. Note that these last three examples show that topologies aren't "monotonic", i.e. if you know $\mathcal{T}_2$ is a topology and $\mathcal{T}_4 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$, you can't conclude that either $\mathcal{T}_4, \mathcal{T}_3$ are necessarily topologies.

2.2 General Families

Let's look at some more general examples.

Definition 7. For any set X, the set $\mathcal{T} = \{\emptyset, X\}$ is a topology called the *trivial topology* or *indiscrete topology*. The proof that this is a topology follows by considering all 4 of the subsets $\mathcal{S} \subseteq \mathcal{T}$ and verifying that their unions/intersections lie in $\mathcal{T}$.

Definition 8. For any set X, the power set $\mathcal{T} = \mathcal{P}(X)$ (i.e. the set of all subsets of X) is a topology called the *discrete topology*. The proof that this is a topology follows from the fact that unions/intersections of subsets of X continue to be subsets of X (and hence lie in $\mathcal{T}$.

Definition 9. For any set X, the set $\mathcal{T} = \{S \subseteq X : |X \setminus S| < \infty\}$ (i.e. the set of elements which contain all but a finite number of points from X) is a topology called the *cofinite topology* or *finite complement topology*. The proof that this is a topology follows from the fact that unions/finite intersections of cofinite sets are cofinite (this requires a bit more of an argument involving De Morgan's laws).

2.3 Euclidean and Subspace Topologies

Here we discuss possible the two most important topologies which should always be at the back of your mind.

Definition 10. For $x_0 \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$, define the *open ball* $B(x_0, \varepsilon) = \{x : |x - x_0| < \varepsilon\}$. For $X = \mathbb{R}^n$, consider the set $\mathcal{T}$ consisting of all sets S such that for all $x_0 \in S$ there exists an open ball $B(x_0, \varepsilon) \subseteq S$. Then $\mathcal{T}$ is a topology called the *Euclidean topology* or *standard topology*.

Throughout this course, whenever we consider $\mathbb{R}^n$, we will assume it is a topological space with the Euclidean topology unless stated otherwise.

We next look at a general way for generating new topologies from old ones.

Definition 11. Given a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, we define the *subspace topology* $\mathcal{T}' = \{U \cap Y : U \in \mathcal{T}\}$. Unless stated otherwise we will always assume subsets $Y \subseteq X$ come equipped with the subspace topology, in which case we say that Y is a *subspace* of X.

Claim 2.2. *The subspace topology is a topology.*

Proof. $\emptyset, X$ are easy. For finite intersections, if you have $V_1, \dots, V_r$ open then $V_i = U_i \cap Y$ for some U_i, then $\bigcap V_i = \bigcap U_i \cap Y$ which is open since X is a topology. The proofs for unions is similar

Actually, the most naive proof for arbitrary unions requires invoking the axiom of choice (this is a very subtle error; it was only noticed in 2018!). Since this is an undergraduate class I'm not going to fret over this, but can talk about it in office hours for those that are interested. □

Example 2.3. Let $X = \mathbb{R}$ with the Euclidean topology and $Y = [0, 1] \subseteq \mathbb{R}$ with the subspace topology. Which of the following sets are open in X? Which are open in Y?

- $(1/4, 3/4)$
- $(1/2, 1]$
- $[1/4, 3/4)$

Around 80% of the topologies we consider in the class will either be $\mathbb{R}^n$ with the Euclidean topology or some subset $X \subseteq \mathbb{R}^n$ equippped with the subspace topology. Here are a few common examples of these sorts of spaces:

- $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ (open ball).
- D^n (open disk)
- S^{n-1} ($(n-1)$-dimensional sphere
- $I^n = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$ (n-dim cube, draw some examples).

Warning: if X is a space and $Y \subseteq X$ is a subspace, it is somewhat ambiguous to talk about "open sets" (do we mean open in X or in Y?). We deal with this as follows.

Definition 12. If $Y \subseteq X$ is a subspace, we say a set A is open in Y if it belongs to the topology of Y, and similarly we define what it means for A to be open in X.

In some situations there's no ambiguity, as in the following.

Claim 2.4. *If $Y \subseteq X$ is an open set in X, then every A which is open in Y is also open in X.*