Intro to Topology

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Part I

Basic Definitions and Examples

1 What is Topology?

This question is somewhat complicated to answer because people use the word "topology" to refer to two related but distinct areas of study:

- *Modern topology* is roughly speaking the study of geometric objects such as spheres, Möbius strips, Klein bottles, and so on. If a mathematician says they "study topology", this is typically what they're referring to.

- *Point set topology* (also called *general topology*) is a very general framework that includes modern topology, much of calculus, and many other areas of mathematics. **This** is what the present course is all about, and from now on whenever I say the word "topology" I'll be referring to this concept.

The central object studied in topology are mathematical objects called *topologies*. So again we're left with the question: what is a topology?

1.1 What is a Topology: the Short Answer

The definition for a topology is a follows. At this point it should **not** be obvious to you why in the world you would ever consider something like this. Here and throughout, given a set X we let $\mathcal{P}(X)$ denote the power set of X, i.e. the set of all subsets of X.

Definition 1. Given a set X, a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* of X if the following conditions are satisfied:

(a) $\emptyset, X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under (arbitrary) unions, i.e. for any $\mathcal{S} \subseteq \mathcal{T}$, we have $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

(c) $\mathcal{T}$ is closed under *finite* intersections, i.e. for any *finite* subset $\mathcal{S} \subseteq \mathcal{T}$, $\bigcup_{U \in \mathcal{S}} U \in \mathcal{T}$.

Again, this is a strange definition that should seem totally bizarre. In the next subsection I'll attempt to provide motivation for why one might possibly come up with this definition. Those that aren't interested/confused by this discussion can completely ignore it without affecting their understanding of the rest of the material in this course.

1.2 What is Topology: the Long Answer

The study of topology originates with the study of calculus/real analysis. When you took courses in these areas, you learned a number of important concepts about the set of real numbers $\mathbb{R}$, as well as about functions f from $\mathbb{R}$ to $\mathbb{R}$. In particular, two very important definitions are:

1. What it means for a sequence of real numbers $(x_n)_{n\geq 1}$ to *converge* to a real number x_0.

2. What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous*.

The central aim of topology is to give a *general framework* which expands these definitions for real numbers to a much broader class of mathematical objects. In particular, it aims to answer the following two questions:

1. What does it mean for a sequence of "objects" $(x_n)_{n\geq 1}$ to *converge* to another object x_0? For example, what does it mean for a sequence of functions $(f_n)_{n\geq 1}$ to converge to another function?

2. What does it mean for a function $f : X \to Y$ between two "nice objects" X, Y to be *continuous*? For example, what does it mean for a map $f : S^2 \to S^2$ from the sphere to itself to be continuous?

We'll postpone the second question and focus on convergence. Of course, any "reasonable" answer should in particular recover the original definition of convergence from real analysis. With this in mind, let's recall this definition and then think about how we might generalize it.

Definition 2. We say that a sequence of real numbers $(x_n)_{n\geq 1}$ *converges* to a real number x_0 if for all $\varepsilon > 0$, there exists an integer N_ε such that $|x_n - x_0| < \varepsilon$ for all $n \geq N_\varepsilon$.

While this is a fine definition, it's a little difficult to generalize. It turns out (for reasons that should not be obvious at this point) that a "better" definition can be made by utilizing the language of *open intervals*, which we recall are sets $I \subseteq \mathbb{R}$ of the form $\{x : a < x < b\}$ for some $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 3. We say that a sequence of real numbers $(x_n)_{n\geq 1}$ *converges* to a real number x_0 if for every open interval I containing x_0, there exists an integer N_I such that $x_n \in I$ for all $n \geq N_I$.

Claim 1.1. *These two definitions are equivalent. That is, a sequence $(x_n)_{n\geq 1}$ and real number x_0 satisfy the conditions of Definition 2 if and only if they satisfy the conditions of Definition 3.*

Definition 3 has several advantages over Definition 2. First, it avoids any mention of the real number ε (which is mathematically nice[1] since we ultimately want to generalize things beyond

[1]This is also psychologically nice for those who have painful memories of ε from real analysis.

real numbers). Second, it frames the definition in terms of the more "geometric" concept of open intervals.

Now, at this point it probably still isn't obvious how to generalize Definition 3 to more general objects (like sequences of functions). However, if you were to spend 30 years about this problem, then perhaps you would come up with the following idea: replace the words "open interval" in Definition 3 with the words "nice set" (where the exact definition of "nice set" depends on your exact problem at hand) and use this as your definition of convergence. Somewhat more precisely, we'll try to work with the following definition (which at this point in time you don't need to memorize since we'll forget about it a moment).

Definition 4. Given a set X, we call any set $\mathcal{T} \subseteq \mathcal{P}(X)$ a *pre-topology*[2] of X. We say that a sequence of points $(x_n)_{n\geq 1}$ with $x_n \in X$ *converges* to a point $x_0 \in X$ with respect to $\mathcal{T}$ if for every $I \in \mathcal{T}$ containing x_0, there exists an integer N_I such that $x_n \in I$ for all $n \geq N_I$.

For example, if $X = \mathbb{R}$ and $\mathcal{T}$ is the set of open intervals of $\mathbb{R}$, then this exactly recovers Definition 3. Here are a few more (extreme) examples to give some more familiarity with these definitions.

Claim 1.2. *Let X be an arbitrary set.*

*(a) If $\mathcal{T} = \emptyset$ (i.e. if $\mathcal{T}$ contains no subsets of X), then **every** sequence of points $(x_n)_{n\geq 1}$ in X converges to **every** point $x_0 \in X$ with respect to $\mathcal{T}$*

(b) If $\mathcal{T} = \mathcal{P}(X)$ (i.e. if $\mathcal{T}$ contains every subset of X), then a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point $x_0 \in X$ with respect to $\mathcal{T}$ if and only if there exists some N such that $x_n = x_0$ for all $n \geq N$ (i.e. iff x_n is "eventually constant").

(c) If $\mathcal{T} = \{\{x\} : x \in X\}$ (i.e. if $\mathcal{T}$ is the set of singletons),then a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point $x_0 \in X$ with respect to $\mathcal{T}$ if and only if there exists some N such that $x_n = x_0$ for all $n \geq N$ (i.e. iff x_n is "eventually constant").

These last two examples suggest the following definition.

Definition 5. Two pre-topologies $\mathcal{T}, \mathcal{T}'$ for the same set X are said to be *equivalent* if: a sequence $(x_n)_{n\geq 1}$ converges to x_0 with respect to $\mathcal{T}$ if and only if it converges to x_0 with respect to $\mathcal{T}'$.

For example, the claim above shows the collection of singletons $\mathcal{T}$ is equivalent to $\mathcal{P}(X)$. Given that these two collections are equivalent to each other, which one should we work with, i.e. which is "better"? A possible answer is that the larger collection $\mathcal{P}(X)$ is "better" because its extra elements give us extra flexibility. This suggests the following problem.

Question 1.3. *Given a pre-topology $\mathcal{T}$, what is the "largest" pre-topology $\mathcal{T}'$ which contains $\mathcal{T}$ and which is equivalent to $\mathcal{T}$?*

[2]This name is not standard at all and we will never use it beyond this first pre-lecture.

This question seems a little daunting, so instead we ask the following weaker question.

Question 1.4. *Given a pre-topology $\mathcal{T}$, are there any "obvious" sets U that we can add to $\mathcal{T}$ so that $\mathcal{T} \cup \{U\}$ is equivalent to $\mathcal{T}$?*

Again if you think about this for 30 years you might realize the following.

Claim 1.5. *Let X be a set, $\mathcal{T}$ a pre-topology, and $\mathcal{S} \subseteq \mathcal{T}$ some non-empty subset of its elements.*

(a) $\mathcal{T} \cup \{\emptyset\}$ is equivalent to $\mathcal{T}$.

(b) $\mathcal{T} \cup \{X\}$ is equivalent to $\mathcal{T}$.

(b) $\mathcal{T} \cup \{\bigcup_{U \in \mathcal{S}} U\}$ is equivalent to $\mathcal{T}$.

(c) If $\mathcal{S}$ is a finite set, then $\mathcal{T} \cup \{\bigcap_{U \in \mathcal{S}} U\}$ is equivalent to $\mathcal{T}$.

(d) Part (c) does not hold in general if $\mathcal{S}$ is allowed to be an infinite set[3].

Sketch of Proof. For (a), since $\emptyset$ contains no elements of X it doesn't affect whether any given element is the limit of a sequence.

For (b), one can always take $N_X = 1$ (since every sequence lies in X for all $n \geq 1$).

For (c), take $N_{\bigcup_{U \in \mathcal{S}} U}$ equal to N_U for any $U \in \mathcal{S}$.

For (d), take $N_{\bigcap_{U \in \mathcal{S}} U} = \max_U N_U$ (note how this requires the set $\mathcal{S}$ to be finite). □

That is, given any pre-topology $\mathcal{T}$, we can freely add in $\emptyset$ and X, as well as (arbitrary) unions and finite intersections of elements of $\mathcal{T}$ into $\mathcal{T}$ to make a (possibly) larger pre-topology which is equivalent to $\mathcal{T}$. In particular, this means that the largest pre-topology $\mathcal{T}'$ which is equivalent to $\mathcal{T}$ must contain $\emptyset, X$ and be "closed" under taking unions and finite intersections. This is exactly the definition of a topology!

2 Definitions and Examples

Let's again restate the definition of a topology, as well as some related definitions that will serve as a useful language for talking about topologies.

Definition 6. Given a set X, a set $\mathcal{T}$ of subsets of X is called a *topology* of X if the following hold:

(a) $\emptyset, X \in \mathcal{T}$.

(b) $\mathcal{T}$ is closed under arbitrary unions. That is, for any subset $\mathcal{S} \subseteq \mathcal{T}$, the set $\bigcup_{U \in \mathcal{S}} U$ is in $\mathcal{T}$.

[3] Hint: take $X = \mathbb{R}$, $\mathcal{T}$ to be the set of open intervals, and $\mathcal{S} = \{(-\frac{1}{n}, \frac{1}{n})\}$.

(c) $\mathcal{T}$ is closed under finite intersections. That is, for any finite subset $\mathcal{S} \subseteq \mathcal{T}$, the set $\bigcap_{U \in \mathcal{S}} U$ is in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*. We will call the pair $(X, \mathcal{T})$ a *topological space*. When $\mathcal{T}$ is clear from context we simply write X instead of $(X, \mathcal{T})$.

Remark 2.1. For arbitrary unions, the book likes to use the notation $\bigcup_{\alpha \in J} U_\alpha$ where J is an "index set", and we will occasionally use this notation in class as well. I recommend looking at Chapter 1 Section 5 of the book to get a more detailed explanation for how this notation is used throughout the book.

Now that we have the definition of a topology in hand, let's pause for a moment and look at some examples and non-examples of topologies.

2.1 Finite Topologies

Is the following pair $(X, \mathcal{T})$ a topological space: In class write all these down and ask students what they think the answers are

$$X = \{a, b, c\},\ \mathcal{T}_1 = \{\{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

No! It fails to have $\emptyset \in \mathcal{T}_1$. Okay, what about

$$X = \{a, b, c\},\ \mathcal{T}_2 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

Well, we have $\emptyset, X \in \mathcal{T}_2$, and it is not difficult to check by hand that this is closed under unions and (finite) intersections (e.g. $\{a, b\} \cap \{b\} = \{b\} \in \mathcal{T}_2$; the slicker way is to note that unions/intersections of sets containing b continue to be sets containing b) so this *is* a topology! What about

$$X = \{a, b, c\},\ \mathcal{T}_3 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}?$$

No! This isn't closed under intersections $\{a, b\} \cap \{a, c\} = \{a\} \notin \mathcal{T}$. What about

$$X = \{a, b, c\},\ \mathcal{T}_4 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}?$$

No! Again not closed under intersections $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_4$. Note that these last three examples show that topologies aren't "monotonic", i.e. if you know $\mathcal{T}_2$ is a topology and $\mathcal{T}_4 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$, you can't conclude that either $\mathcal{T}_4, \mathcal{T}_3$ are necessarily topologies.

2.2 General Families

Let's look at some more general examples.

Definition 7. For any set X, the set $\mathcal{T} = \{\emptyset, X\}$ is a topology called the *trivial topology* or *indiscrete topology*. The proof that this is a topology follows by considering all 4 of the subsets $\mathcal{S} \subseteq \mathcal{T}$ and verifying that their unions/intersections lie in $\mathcal{T}$.

Definition 8. For any set X, the power set $\mathcal{T} = \mathcal{P}(X)$ (i.e. the set of all subsets of X) is a topology called the *discrete topology*. The proof that this is a topology follows from the fact that unions/intersections of subsets of X continue to be subsets of X (and hence lie in $\mathcal{T}$.

Definition 9. For any set X, the set $\mathcal{T} = \{S \subseteq X : |X \setminus S| < \infty\} \cup \{\emptyset\}$ (i.e. the set of elements which contain all but a finite number of points from X) is a topology called the *cofinite topology* or *finite complement topology*. The proof that this is a topology follows from the fact that unions/finite intersections of cofinite sets are cofinite (this requires a bit more of an argument involving De Morgan's laws).

2.3 Euclidean and Subspace Topologies

Here we discuss possible the two most important topologies which should always be at the back of your mind.

Definition 10. For $x_0 \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}$, define the *open ball* $B(x_0, \varepsilon) = \{x : |x - x_0| < \varepsilon\}$. For $X = \mathbb{R}^n$, consider the set $\mathcal{T}$ consisting of all sets S such that for all $x_0 \in S$ there exists an open ball $B(x_0, \varepsilon) \subseteq S$. Then $\mathcal{T}$ is a topology called the *Euclidean topology* or *standard topology*.

Draw a picture of an open set in $\mathbb{R}^2$.

Throughout this course, whenever we consider $\mathbb{R}^n$, we will assume it is a topological space with the Euclidean topology unless stated otherwise.

We next look at a general way for generating new topologies from old ones.

Definition 11. Given a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, we define the *subspace topology* $\mathcal{T}' = \{U \cap Y : U \in \mathcal{T}\}$. Unless stated otherwise we will always assume subsets $Y \subseteq X$ come equipped with the subspace topology, in which case we say that Y is a *subspace* of X.

Claim 2.2. *The subspace topology is a topology.*

Proof. $\emptyset, X$ are easy. For finite intersections, if you have $V_1, \ldots, V_r$ open then $V_i = U_i \cap Y$ for some U_i, then $\bigcap V_i = \bigcap U_i \cap Y$ which is open since X is a topology. The proofs for unions is similar

Actually, the most naive proof for arbitrary unions requires invoking the axiom of choice (this is a very subtle error; it was only noticed in 2018!). Since this is an undergraduate class I'm not going to fret over this, but can talk about it in office hours for those that are interested. □

Example 2.3. Let $X = \mathbb{R}$ with the Euclidean topology and $Y = [0, 1] \subseteq \mathbb{R}$ with the subspace topology. Which of the following sets are open in X? Which are open in Y?

- $(1/4, 3/4)$
- $(1/2, 1]$
- $[1/4, 3/4)$

Around 80% of the topologies we consider in the class will either be $\mathbb{R}^n$ with the Euclidean topology or some subset $X \subseteq \mathbb{R}^n$ equippped with the subspace topology. Here are a few common examples of these sorts of spaces:

- $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ (open ball).
- D^n (open disk)
- S^{n-1} ($(n-1)$-dimensional sphere
- $I^n = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$ (n-dim cube, draw some examples).

Warning: if X is a space and $Y \subseteq X$ is a subspace, it is somewhat ambiguous to talk about "open sets" (do we mean open in X or in Y?). We deal with this as follows.

Definition 12. If $Y \subseteq X$ is a subspace, we say a set A is open in Y if it belongs to the topology of Y, and similarly we define what it means for A to be open in X.

In some situations there's no ambiguity, as in the following.

Claim 2.4. *If $Y \subseteq X$ is an open set in X, then every A which is open in Y is also open in X (intersection of open sets is open).*

3 Closed Sets

- Recap: topologies (and that given a topological space $(X, \mathcal{T})$, a set $U \subseteq X$ is called open if $U \in \mathcal{T}$), examples (Euclidean, subspace)

- Idea with definition of "open" is that these generalize notion of open sets from real analysis, but equally important in real analysis is closed sets (e.g. intermediate/extreme value theorem both involve functions $f : [a, b] \to \mathbb{R}$, i.e. both involve closed intervals).

  So, what's the right way to define closed sets for general topologies?

- Definition: a set $A \subseteq X$ is called *closed* if its complement $X - A$ is open.

- Warmup: which of the following sets A are open/closed in $X = \mathbb{R}$?

  - $[0, 1]$

  - $\mathbb{R}_{>0}$

  - $\mathbb{Q}$

  - $\mathbb{R}$

  - $\emptyset$

- Def: a set $A \subseteq X$ which is both open and closed is called clopen.

- Eg in discrete topology, every set is open/closed/clopen.

- Thm: if X is a topological space then: (1) $\emptyset, X$ are closed, (2) arbitrary intersections of closed sets are closed, (3) finite unions of closed sets are closed.

  With this we see we could have defined topology via closed sets and gotten same theory; there's no real distinction.

- As in the previous lecture, if we have a subspace $Y \subseteq X$ it can be ambiguous to say A is closed. In this case we will say that A is closed in Y or closed in X as apporpriate.

  Claim: if $Y \subseteq X$ is closed, then any A which is closed in Y is closed in X.

- Given $A \subseteq X$ there's two important sets we can associate to it. Def: the interior A^o (or $intA$) is the union of all the open sets contained in A, and the closure $\bar{A}$ is intersection of closed sets containing it.

- E.g intuitively what is $[0, 1)$ interior/closure?

- Essentially, A^o is the largest open subset of A (and similar $\bar{A}$).

- Prop: Prove these
  - $A^o, \bar{A}$ are open/closed.
  - $A^o \subseteq A \subseteq \bar{A}$.
  - If A is open then $A = A^o$, and if A is closed then $A = \bar{A}$.

- In order to characterize closure we need some definitions: if U is an open set containing x, then we say that U is a *neighborhood* of x. We say that two sets A, B intersect if $A \cap B \neq \emptyset$.

- Thm: $x \in \bar{A}$ iff every neighborhood of x intersects A.
  - Equivalent to prove $x \notin \bar{A}$ iff exists neighborhood disjoint from A. Indeed, if $x \notin \bar{A}$ then there exists closed set C containing A but not x, then $X - C$ is a neighborhood disjoint from A. Reverse direction similar.

- Examples for subsets of $\mathbb{R}$:
  - $A = \{n^{-1}\}$, closure is this plus 0 (easy to see 0 is in due to neighborhood description, everything else has neighborhood disjoint from it).
  - $A = \mathbb{Q}$, closure is $\mathbb{R}$.

- Neighborhoods are one useful way to characterize closures. Another way is through limit points. Definition: given a subset $A \subseteq X$, a point x is called a *limit point* (or cluster point, or point of accumulation) of A if every neighborhood of x intersects A in some point *other than itself*. Equivalently, x is a limit point if $x \in \overline{A - \{x\}}$.

- E.g. for $X = \mathbb{R}$, $A = \{0\}$ no point is a limit point. If $A = (0, 1]$ every point in $[0, 1]$ is a limit point. If $A = \{n^{-1}\}$ then only 0 is a limit point.

  Intuitively, x is a limit point if there's a sequence in $A - x$ "converging" to x.

- Thm: if A' denotes the set of limit points of A, then $\bar{A} = A \cup A'$.
  - If $x \in A'$ then every neighborhood intersects A, so by the one theorem it's in the closure so $A \cup A' \subseteq \bar{A}$.
  - If $x \in \bar{A} \setminus A$, then every neighborhood intersects A, and necessarily $A - x$ since $x \notin A$, so $x \in A'$.

- Corollary: a set A is closed if it contains all of its limit points ($A = A \cup A'$ implies $A' \subseteq A$).

4 Convergent Sequences and Hausdorff Spaces

- Again the goal of topology is to generalize concepts from real analysis, and now that we have a lot of examples/terminology, we can finally start defining these analogs.

- One important concept that we've seen is convergent sequences. Definition: given a topological space X, we say that a sequence of points $(x_n)_{n\geq 1}$ in X converges to a point x if for all neighborhoods U of x, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

- This definition can be used to motivate the name "limit point" from last time.

  Prop: let X be a space and $A \subseteq X$. If $x \in X$ is such that there exists a sequence $(x_n)_{\geq n}$ in $A - x$ which converges to x, then x is a limit point of A.

  - Proof is that for any neighborhood there exist infinitely many x_n in it, all of which are in A and all of which are distinct from x.

  - Converse turns out to be false (i.e. there exist limit points which are not limits) but it's not super easy to construct; see this. This is true however in nice spaces (e.g. metric spaces).

- Natural question that pops up when playing with sequences: for every sequence $(x_n)_{n\geq 1}$, does there exist at most one point x which the sequence converges to?

  Intuition with $\mathbb{R}^n$ says, yes, but this is false: in trivial topology every sequence converges to every point.

- This is a weird situation we'd like to avoid.

  Definition: a topological space is said to be *Hausdorff* or T_2 if for each pair of distinct points x, y, there exist neighborhoods U, V of x, y respectively which are disjoint. Draw picture

- Thm: if X is Hausdorff, then every sequence converges to at most one point.

  - Assume for contradiction converge to x, y let U, V be neighborhoods. Take N_U, means all $n \geq N_U$ lie in U i.e. aren't in V, contradicting the existence of N_V.

  - Note that the definition of Hausdorff is designed to be essentially the weakest condition such that this property holds.

- Examples: proofs

  - $\mathbb{R}^n$ is Hausdorff

  - Trivial with at least two points is not

  - Discrete is Hausdorff

  - Finite complement with an infinite number of points is not (every two open sets intersect in all but finitely many points).

- This next proof will use the following trick that will be used many times throughout this course:

  Neighborhood trick: a set $U \subseteq X$ is open iff for every $x \in U$ there exists an open set V_x with $x \in V_x \subseteq U$. (Proof if U is open is easy, other direction uses unions).

- Already saw Hausdorff is nice because sequences have at most one limit, which agrees with our intuition from $\mathbb{R}^n$. It also plays nicely with intuition for closed sets.

  Thm: if X is Hausdorff, then every finite subset $A \subseteq X$ is closed.

  - Not true in general: $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ doesn't have b closed.
  - Proof: suffices to prove it when $A = \{x\}$ (since unions of closed sets are closed), i.e. that $X - x$ is open. Because X is Hausdorff, each $y \in X - x$ has a neighborhood U_y disjoint from x. Hence $\bigcup U_y = X - x$ is open.
  - Note that we didn't need the full power of Hausdorff in this proof: we only used that each y has a neighborhood disjoint from each x (so e.g. it holds for finite complement topology). This disjoint neighborhood condition is called the T1 condition; we'll return to this in Chapter 4.

5 Continuous Functions

- Recap: convergence, Hausdorff (and sequences converge to at most one point).
- Again, one of the main points of topology is to generalize key concepts from real analysis.
- Recall form calculus that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for all $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exist $\delta > 0$ such that for all x with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$.

  Phew, that's a mouthful.
- Def: a map $f : X \to Y$ between two topological spaces is called *continuous* if for every open set U in Y, the set $f^{-1}(U)$ is open in X.
  - That is f is continuous if the pre-image of open sets are open.
  - Warning: The notation $f^{-1}(U) := \{x : f(x) \in U\}$ is the *pre-image* of f NOT the inverse of f (which may not exist).
- Claim: this is equivalent to the calculus definition for the Euclidean topology.

5.1 Examples

- Eg take $X = \{a, b\}$, $\mathcal{T}_x = \{\emptyset, X, \{a\}\}$, define $Y = \{1, 2\}$ with essentially the same topology. Consider all 4 possible maps $f : X \to Y$ and ask which are continuous (all but $f(a) = 2, f(b) = 1$ because $f^{-1}(1) = b$ which isn't open).
- Prop: if Y has the trivial topology, then every map $f : X \to Y$ is continuous.

  "Most" maps from trivial topology on X aren't continuous (requires every open set of Y to contain $f(X)$ or be empty).
- Prop: if X has the discrete topology, then every map $f : X \to Y$ is continuous.

  "Most" maps from discrete Y aren't continuous.
- If $\mathcal{T}, \mathcal{T}'$ are topologies on the same set X, when is the identity map $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ with $f(x) = x$ continuous? Ans: when $\mathcal{T}' \subseteq \mathcal{T}$.

  Def if $\mathcal{T}' \subseteq \mathcal{T}$ then we say that $\mathcal{T}'$ is coarser than $\mathcal{T}$ and that $\mathcal{T}$ is finer than $\mathcal{T}'$.

  Prop: $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ with f the identity map $f(x) = x$ is continuous iff $\mathcal{T}$ is finer than $\mathcal{T}'$.
- Prop: if $A \subseteq X$ is given the subspace topology, then the inclusion map $\iota : A \to X$ defined by $\iota(a) = a$ is continuous.
  - $f^{-1}(U) = U \cap A$, which is open in A by construction of subspace topology.

- Aside: this proof shows that the subspace topology is the "weakest" topology we can put on $A \subseteq X$ so that the inclusion map is continuous.

  General theme: if you have a "natural map" $f : X \to Y$, then you should define the "weakest" topologies on X, Y such that f is continuous (eg subspace above).

- The function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0$ is continuous (two ways, (1) directly and (2) because calculus).

- Warning: f continuous does NOT mean it maps open sets to open sets. E.g. the previous example.

5.2 Equivalences and Constructions

- Aside: why would you ever come up with the definition of continuity?

  - Intuition of $\varepsilon - \delta$ definition definition from calculus: small changes to your input lead to small changes in output.

  - More precisely, we say that $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if for any "tolerance" $\varepsilon > 0$ we can find $\delta > 0$ sufficiently small so that $f((x-\delta, x+\delta)) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$.

  - If we tried to generalize this intuition, we might come up with the following definition:
    End of aside

- Def: a map $f : X \to Y$ is said to be continuous at $x \in X$ if for every open set $f(x) \in V \subseteq Y$, there exists an open set $x \in U \subseteq X$ such that $f(U) \subseteq V$.

  Prop: a map $f : X \to Y$ is continuous (as defined at the start of class) iff it is continuous at every point $x \in X$ (as defined above).

  - Assume f is continuous and you have some $f(x) \in V \subseteq Y$, what U should you define to have $f(U) \subseteq V$? Take $U = f^{-1}(V)$; this works by construction.

  - Assume f is continuous at each point. Let V be open and $x \in f^{-1}(V)$. Since $f(x) \in V$, exists some neighborhood U_x with $f(U_x) \subseteq V$, and hence $U_x \subseteq f^{-1}(V)$. Note that $f^{-1}(V) = \bigcup U_x$, so it's open.

- Because continuity is such a fundamental concept, it will be useful to have a few more equivalent formulations.

  Thm: X, Y be topological spaces and $f : X \to Y$. TFAE:

  1. f is continuous (i.e. preimage of open sets are open)

  2. For every closed set $B \subseteq Y$, the set $f^{-1}(B)$ is closed in X.

  3. For every subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A))}$

Proof:

- (1) to (2): assume f is continuous and let B be closed in Y. Taking $V = Y - B$, basic set theory says Then $f^{-1}(V) = X \setminus f^{-1}(B)$. Since V is open, this set is open, which means $f^{-1}(B)$ is the complement of an open set and hence open. Other direction is basically the same.

- (1) to (3): Assume f continuous and $A \subseteq X$. Aim to show $x \in \overline{A}$ implies $f(x) \in \overline{f(A)}$. Let V be neighborhood of $f(x)$, pre-image is open so neighborhood of x, thus intersects A, so $f(f^{-1}(V)) \subseteq V$ intersects $f(A)$. Since every neighborhood of $f(x)$ intersects $f(A)$, we conclude $f(x) \in \overline{f(A)}$.

- (3) to (2): Let B be closed in Y and take $A = f^{-1}(B)$; aim is to show $A = \overline{A}$. Note that $f(A) = f(f^{-1}(B)) \subseteq B$. Thus if $x \in \overline{A}$, $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$, which means $x \in f^{-1}(B) = A$. Thus $\overline{A} \subseteq A$ and they must equal each other.

- We now look at some ways of constructing new continuous functions from old ones.

  Prop: if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

- For this next result we'll want the following which we've mentioned a few times already now: let $A \subseteq Y \subseteq X$. (a) If A is open in Y and Y is open in X, then A is open in X. (b) same with closed. Prove open one

- (Pasting Lemma) Let $X = A \cup B$ with A, B either both open or both closed in X and let $f : A \to Y$ and $g : B \to Y$ be continuous maps that agree at their intersection, i.e. $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$ is continuous.

  - E.g. the function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x$ for $x \leq 0$ and $h(x) = x/2$ for $x \geq 0$ is continuous because of this result.

  - Result is false if one of A, B is open and the other closed, e.g. $X = \mathbb{R}$, $A = (-\infty, 0]$ and $B = (0, \infty)$ with $f(x) = -1$ and $g(x) = 1$.

  - Proof: only prove case when A, B both closed. Let C be a closed set. Not difficult to argue $h^{-1}(C) = f^{-1}(V) \cup g^{-1}(C)$. Since f, g continuous, these two sets are *closed in* A, B. Since A, B are closed in X, the lemma above implies these two sets are closed in X. Thus intersection is closed, proving the result by equivalent formulation of continuity.

5.3 Homeomorphisms

- Let $X = \{a, b\}$, $\mathcal{T}_x = \{\emptyset, X, \{a\}\}$ and similarly define $Y = \{1, 2\}$ with $\mathcal{T}_y$ basically the same. Ask if $(X, \mathcal{T}_x) = (Y, \mathcal{T}_y)$? Answer is no, but they are "equivalent".

- Definition: a map $f : X \to Y$ is said to be a *homeomorhism* if (a) f is a bijection, (b) f is continuous, and (c) f^{-1} is continuous (this exists because f is a bijection); equivalently $f(U)$ is open whenever U is open.
  - If there exists a homeomorphism between X, Y we say these spaces are *homeomorphic* and write $X \cong Y$.
  - Note for those familiar with algebra that although this sounds like "homomorphism" its much closer to isomorphism.
- Eg are the X, Y at the start of this subsection homeomorphic?
  - What homeomorphism shows this?
  - Check that that this works: draw a column on the right listing the open sets of Y with the open sets of X on the other side, draw arrows from Y backwards labeled f^{-1} to their corresponding sets, then arrows going the other way labeled f.
  - Aside: a map f being a homeomorphism is equivalent to saying it "induces" a bijection between $\mathcal{T}_x$ and $\mathcal{T}_y$ (as the example above demonstrates), i.e. that the two topologies are just "relabelings" of each other. This relabeling definition is perhaps more intuitive, but the homeomrphism definition is easier to work with in practice.
- Eg let $B_n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ denote the ball of radius r centered at x.

  Claim: $B_n(x, r) \cong B_n(x + a, r)$ for all a (i.e. translates of the same space are homeomorphic)
  - What's the homeomorphism? $f(z) = z + a$.
  - That f is a bijection is straightforward.
  - That f and its inverse $g(z) = z - a$ are continuous follows "from calculus" (i.e. we know the topological definition of continuity is equivalent to the calculus definition, and we know from real analysis that translations are continuous functions).
  - Convention: throughout this course, if you have a function $f : X \to Y$ with X, Y subspaces of Euclidean space, you are allowed to say f is continuous "by calculus" whenever it follows from basic real analysis that f is continuous.
  - Aside: one can also prove f is continuous by hand (which is what I originally planned to do), but it is a real pain. The proof will become a lot easier once we have the tools from next lecture. Maybe sketch this out.
- Claim: if $r, c > 0$, then $B_n(0, r) \cong B_n(0, cr)$ (i.e. dilates of the same space are homeomorphic). Proof: $f(x) = cx$ is a homeomorphism "by calculus".

- The two statements above imply that any two balls in $\mathbb{R}^n$ of finite radius are homeomorphic to each other. In fact, this continues to hold even for infinite radiuses:

  Claim: $B_n(0,1) \cong \mathbb{R}^n$. Proof: take $f(x) = \frac{x}{1-|x|}$ draw picture of arrows going out, with arrows expanding more farther away, this and its inverse $g(y) = \frac{y}{1+|y|}$ are continuous "by calculus".

- Variant: $[0,1)$ and $\mathbb{R}_{\geq 0}$ with subspace topologies are homoemorphic ($f : [0,1) \to \mathbb{R}_{\geq 0}$ with $f(x) = \frac{x}{1-x}$ is continuous by calculus, its inverse $g(y) = \frac{y}{y+1}$ also continuous by calculus).

- Prop: $X = S^1 = \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 = 1\}$ and $Y = \{(x,y) \in \mathbb{R}^2 : \max(|x|,|y|) = 1\}$ (square) are homeomorphic draw picture, map is $f(x,y) = (\frac{x}{\max(|x|,(|y|)}, \frac{y}{\max(|x|,|y|)})$ and inverse $g(x,y) = (\frac{x}{\sqrt{|x|^2+|y|^2}}, \frac{y}{\sqrt{|x|^2+|y|^2}})$

- More generally, any two subspaces of $\mathbb{R}^n$ are homeomorphic if you can can "twist/bend" one into the other.

  - E.g. S^1 and some wild non-intersecting looking curve.

  - E.g. donut and coffee cup.

- Warning: f being continuous and bijective doesn't imply inverse is continuous, e.g. $[0,1)$ to circle via $f(x) = (\cos(2\pi x), \sin(2\pi x))$ (is a continuous bijection, but its inverse isn't continuous because of preimages around 0)

- Sometimes a map can be a "local" homeomorphism. Definition: let $f : X \to Y$ be an injective continuous map. If the restricted map $f' : X \to f(X)$ is a homeomorphism, then we say that the original map $f : X \to Y$ is an *imbedding*.

- Prop: the relation of being homeomorphic is an equivalence relation.

- Aside: say that a property is a *topological property* if the property is preserved under homeomorphisms.

  - E.g. Cardinality (if have two homeomorphic spaces then necessarily same cardinality because f bijection).

  - E.g. connectedness (see later).

  - Non-e.g.: location (translations), size, boundedness

  - Non-e.g: "smoothness" (e.g. can't distinguish circle vs square). That is, topology is too loose to understand curvature, but this can be resolved through "differential topology".

5.4 Padding for Time

- Various general continuous maps:
  - Constant functions.
  - Restricting domain.
  - Expanding codomain.
- Aside: the two most important definitions in any field of math is (1) what are the objects of study, (2) what are the "nice maps" between these objects? E.g. topological spaces/continuous, vector spaces/linear, sets/functions, groups/homomorphisms. More generally category theory.

6 Basis

- Problem: it can be hard to show that relatively simple maps f are continuous "by hand", e.g. showing the translation $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = x + a$ is tricky.

- Part of the difficulty above is that the definition of open sets in $\mathbb{R}^n$ is complicated: recall that a say U is open in $\mathbb{R}^n$ iff for every $x \in U$ there exists a ball $B_x \subseteq U$ containing x; this means weird shapes can be open draw one.

  Observation: general open sets $U \subseteq \mathbb{R}^n$ can be complex, but they're made up of simple building blocks (i.e. balls). Can we extend this idea?

- Idea: given a collection of sets $\mathcal{B}$, we want to define a topology $\mathcal{T}$ by having $U \in \mathcal{T}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$ (by the neighborhood trick, this is the same as saying every open set is the union of elements of $\mathcal{B}$).

  Problem: $\mathcal{T}$ won't be a topology for arbitrary sets $\mathcal{B}$, so we need to figure out some conditions on $\mathcal{B}$ which makes this work out.

- Definition: given a set X, a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* of X if (1) for every $x \in X$, there exists some $B \in \mathcal{B}$ containing x and (2) for all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$.

  - Eg open balls in Euclidean topology (if distance from x to x_i is d_i, then you can take $B = B(x, \min\{\varepsilon_i - d_i\})$

  - Eg $X = \mathbb{R}^2$ and $\mathcal{R} = \{(a, b) \times (c, d)\} \subseteq \mathbb{R}^2$ (intersection itself is open rectangle).

  - Eg $X = \mathbb{R}$ and $\mathcal{H} = \{[a, b)\}$ (again intersection just works).

  - $\mathcal{D} = \{\{x\} : x \in X\}$ always works.

- Definition: if $\mathcal{B}$ is a basis for X, the *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ is defined by having $U \in \mathcal{T}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$ (note that this implies $\mathcal{B}$ are all open sets in $\mathcal{T}$).

  - Note this recovers Euclidean if $\mathcal{B}$ is open balls.

  - This is a topology: $\emptyset$ easy, X by (1). Pairwise intersection: there exists $B_i \subseteq U_i$ containing x each time, take intersection, by (2) there's some B contained in this containing x. Arbitrary union, take any i with $x \in U_i$ and then its corresponding basis element.

  - Aside: conditions (1) and (2) for $\mathcal{B}$ being a basis turn out to be equivalent to the condition that $\mathcal{T}$ is a topology, so this really is the "right" definition for a basis to make.

- What topologies do previous examples generate? Claim (will see soon) $\mathcal{R}$ generates Euclidean, i.e. same as balls $\mathcal{B}$ (despite the two having no elements in common). Topology generated by $\mathcal{H}$ is something other than Euclidean called "lower limit topology". $\mathcal{D}$ is discrete.

- The exact definition of the topology generated by $\mathcal{B}$ is somewhat complicated. Here's a cleaner formulation. Lemma: if $\mathcal{B}$ is a basis, then the topology $\mathcal{T}$ generated by $\mathcal{B}$ equals the set of all possible unions of elements of $\mathcal{B}$ (this includes the empty union).

  Proof: Note that $\mathcal{B} \subseteq \mathcal{T}$, and because $\mathcal{T}$ is a topology, it necessarily contains all possible unions of $\mathcal{B}$. Conversely, if $U \in \mathcal{T}$ then for each $x \in U$ there exists $B_x \in \mathcal{B}$ containing x and contained in U, so $U = \bigcup_{x \in U} B_x$.

- Now we get to one of the most useful consequences of basis.

  Thm: if Y is generated by a basis $\mathcal{B}$, then $f : X \to Y$ is continuous iff $f^{-1}(B)$ is open for $B \in \mathcal{B}$.

  - Continuous implies this condition.
  - This condition plus U equal to union of basis elements gives other direction.

- E.g. to check that the translation map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = x + a$ is continuous, it suffices to prove that $f^{-1}(B)$ is open whenever B is an open ball, and this holds since $f^{-1}(B)$ is an open ball.

- Basis play nicely with subspaces.

  Prop: if $\mathcal{B}$ is a basis for X, then $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for Y. (every element of $Y \subseteq X$ is in some element, if you look at the intersection, lift to X, then project back down you get the thing).

- Basis make it easier to check if sets are closed.

  Thm: if X has a basis, then $x \in \bar{A}$ iff every basis element B containing x intersects A.

  - Recall: $x \in \bar{A}$ iff every neighborhood of x intersects A.
  - It suffices to show this latter condition is equivalent to having every neighborhood intersect A, easy because B is open and because neighborhoods always contain a basis sub-neighborhood.

- We know how to go from basis to topology. Sometimes it will be useful to go the other way, i.e. given a topology $\mathcal{T}$ how do we find a basis for it?

  - Prop: let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ a family of subsets of X. If (a) every element of $\mathcal{B}$ is open and (b) For every open set $U \subseteq X$ and every $x \in U$ there is an element $B \in \mathcal{B}$ with $x \in B \subseteq U$, then $\mathcal{B}$ is a basis which generate the topology on X.

- Proof (if $\mathcal{B}$ satisfies these conditions then it is a basis): Taking $U = X$ implies (1) of basis. Since $\mathcal{B}$ are open sets, $B_1 \cap B_2$ is open so can find a B to satisfy (2), so this is a basis.

- Proof (that the topology generates $\mathcal{T}$): Let $\mathcal{T}'$ be the topology generated by $\mathcal{B}$. Every element $W \in \mathcal{T}'$ is a union of elements of $\mathcal{B}$ which are open sets, so $W \in \mathcal{T}$. On the other hand, each $U \in \mathcal{T}$ can be written as the union of basis elements so $U \in \mathcal{T}'$.

- Corollary: $\mathcal{R}$ generates $\mathbb{R}^2$.

- Basis requires two relatively weak conditions, but sometimes its useful to relax even these.

  - Def: a set $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a *sub-basis* (or pre-basis) if for every $x \in X$, there exists some $B \in \mathcal{B}$ containing x (so it has (1) of the definition of the basis but not necessarily (2)).

  - Claim: the set of finite intersections of a sub-basis $\mathcal{S}$ is a basis. We define the topology generated by $\mathcal{S}$ to be the topology generated by this basis.

  - Claim: if Y is generated by a subbasis $\mathcal{S}$, then $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open for all $U \in \mathcal{S}$.

7 Product Topologies

7.1 Finite Products

- The next few lectures explore forming new topologies by performing "operations" on old ones.

- Definition: given topological spaces X, Y, define the *product topology* on $X \times Y$ as the topology generated by the basis $\mathcal{B} = \{U \times V : U \text{ open in } X, V \text{ open in } Y\}$.

  Claim: $\mathcal{B}$ is a basis (and hence does indeed generate a topology).

- Warning $\mathcal{B}$ is *not* a topology, i.e. the open sets of $X \times Y$ are *not* (the only) elements of $\mathcal{B}$. Insert picture of union of two rectangles

- Claim: if $\mathcal{B}, \mathcal{C}$ are basis that generate the topologies for X, Y, then $\mathcal{D} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$ generates the product topology $X \times Y$.

  - Recall lemma from before: a set $\mathcal{D}$ is a basis generating a topology $\mathcal{T}$ if (1) $\mathcal{D}$ is a set of open sets, (2) for every $x \in U \in \mathcal{T}$ there exists $x \in D \subseteq U$.

- Cor: if $\mathbb{R}$ has the standard topology, then the product topology $\mathbb{R} \times \mathbb{R}$ equals the standard topology on $\mathbb{R}^2$ (equivalently, the topology generated by the basis of open rectangles is the same as the topology generated by open balls).

- Subspace and product topologies "commute": if you take $X \times Y$ and give $A \times B$ the subspace topology, it's the same as if you first gave A, B the subspace topology and then took the product topology.

- Examples of products:

  - $S^1 \times I^1$ is cyllinder.

  - $S^1 \times S^1$ is torus.

  - $S^1 \times D^2$ is solid torus.

- Def: the projection map $\pi_1 : X \times Y \to X$ has $\pi_1(x, y) = x$, and we similarly define π_2.

  These are the most important maps related to product topologies. In particular, they can be used to generate the product topology.

  Claim: the sets $\{\pi_1^{-1}(U) : U \in \mathcal{T}_X\} \cup \{\pi_2^{-1}(V) : V \in \mathcal{T}_Y\}$, i.e. the sets $U \times Y$ and $X \times V$ form a subbasis for the product topology $X \times Y$ (Pf: every basis element is the intersection of two subbasis elements, every finite intersection of subbasis elements gives you a basis element)

- Prop: the projection maps π_i are continuous (proof: $\pi_1^{-1}(U) = U \times Y$)

  In fact, the product topology on $X \times Y$ is the "weakest" topology such that the projection maps are continuous (analogous to how subspace topology was the weakest so that the inclusion map $\iota : A \to X$ was continuous). This is essentially why we defined things this way.

- Thm: let A be a space and $f : A \to X_1 \times X_2$ a function, say with $f(a) = (f_1(a), f_2(a))$ for some $f_i : A \to X_i$. If $X_1 \times X_2$ is given the product topology, then f is continuous iff f_1, f_2 are both continuous (if f_i are not continuous then easy to construct set, if they are both continuous then suffices to check basis elements, $U \times V$ whose preimage is $f^{-1}(U) \cap g^{-1}(V)$).

- Aside: this is the "universal property" of products (in the sense of category theory).

- Warning: no useful way for saying that a function $h : A \times B \to X$ is continuous.

7.2 General Products

- How do you define product topology for three spaces? Same basic idea, also works for any finite number of products. What about infinite products? This is more subtle.

- First we need to figure out how to define infinite products.

  - Given sets J, X, we define a J-tuple of elements of X to be a function $\mathbf{x} : J \to X$. If $\alpha \in J$ we often denote the value $\mathbf{x}(\alpha)$ by x_α and denote $\mathbf{x}$ by the symbol $(x_\alpha)_{\alpha \in J}$.

  - Given an indexed family of sets $\{A_\alpha\}_{\alpha \in J}$, we define the cartesian product $\prod_{\alpha \in J} A_\alpha$ to be the set of all J-tuples of $X = \bigcup A_\alpha$ such that $x_\alpha \in A_\alpha$ for all $\alpha \in J$.

    If $A_\alpha = X$ for all $\alpha \in J$, then we will write this product as X^J (equivalent to set of all functions from J to X), and if $J = \mathbb{Z}_{>0}$ we use the shorthand X^ω.

  - Eg if $J = \{1, 2\}$ then $A_1 \times A_2$ consists of all functions $x : \{1, 2\} \to A_1 \cup A_2$ with $x_1 \in A_1$ and $x_2 \in A_2$. This is just the usual definition.

  - Eg if $J = \mathbb{Z}_{>0}$ and $A_\alpha = \mathbb{R}$ for all α what is $\prod A_\alpha = \mathbb{R}^{\mathbb{Z}_{>0}} = \mathbb{R}^\omega$? Formally this is all functions $f : \mathbb{Z}_{>0} \to \mathbb{R}$, which (in tuple notation) is the set of sequences of real numbers (e.g. $(n^2)_{n \geq 1}$ is in this set).

  - Most examples of cartesian product will be of the form X^ω, i.e. this will just be infinite sequences.

- Topology? Naive attempt: given a family of topological spaces $\{X_\alpha\}_{\alpha \in J}$ we define the *box topology* on $\prod X_\alpha$ as having the basis consisting of sets $\prod U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open.

  - Lem: this is a basis (and hence a well defined topology).

- This is the simplest thing to do, but it has some serious issues. In particular, nice properties that held for finite products don't hold in general here.
- Define $f : \mathbb{R} \to \mathbb{R}^\omega$ via $f(t) = (t, t, \ldots)$. Simple map that's continuous in each coordinate, but f is not continuous (take preimage of $(-1,1) \times (-1/2, 1/2) \times \cdots$, get $\{0\}$).
- This isn't ideal: we'd like to say like before that if we have a map $f : Y \to X_1 \times X_2 \cdots$ which is continuous in each coordinate then f is continuous.

- Non-obvious solution: given a family of topological spaces $\{X_\alpha\}_{\alpha \in J}$ we define the *product topology* on $\prod X_\alpha$ as having the basis consisting of sets $\prod U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open **and** where $U_\alpha = X_\alpha$ for all but finitely many α.
  - Lem: this is a basis (and hence a well defined topology).
  - Eg for $\mathbb{R}^\omega$, the set $(-1,1) \times (-1/2, 1/2) \times \cdots$ is open in the box topology (since it's a basis element), but it is **not** open in the product topology (since every non-empty open set must contain a basis element)
  - Note that the box and product agree on finite products but for infinite one's the product topology is coarser than box.
  - Product topology, while less obvious, turns out to be way nicer, so from now on whenever we look at infinite product spaces we'll assume they have the product topology unless stated otherwise. One of main reasons is the following.

- How might you come up with this definition? One way is through projection maps $\pi_\alpha : \prod X_\beta \to X_\alpha$ defined by $\pi_\alpha(x) = x_\alpha$.

  Claim: the sets $\bigcup_\alpha \{\pi_\alpha^{-1}(U) : U \in \mathcal{T}_{X_\alpha}\}$ form a subbasis for the product topology $\prod X_\alpha$ (i.e. their finite intersections are exactly the basis elements of the product topology).

  Claim: the product topology is the "weakest" topology such that each projection map π_α are continuous.

- Thm: let $\{X_\alpha\}_{\alpha \in J}$ be a family of spaces, A a space, $f_\alpha : A \to X_\alpha$ a family of functions, and $f : A \to \prod X_\alpha$ defined by $f(a) = (f_\alpha(a))_{\alpha \in J}$. If $\prod X_\alpha$ is given the product topology, then f is continuous iff f_α is continuous for all α.
  - Proof: if some f_α is not continuous then $f_\alpha^{-1}(U_\alpha)$ is not open in A for some open set $U_\alpha \subseteq X_\alpha$. Note that the set $U = \prod U_\beta$ with $U_\beta = X_\beta$ for $\beta \neq \alpha$ is open (since it's a basis element) and

    $$f^{-1}(U) = \bigcap f^{-1}(U_\beta) = f^{-1}(U_\alpha) \bigcap_{\beta \neq \alpha} A = f^{-1}(U_\alpha)$$

    which isn't open by assumption.

- Assume now each f_α is continuous, and recall to prove f is continuous it suffices to show $f^{-1}(U)$ is open for all basis elements $U = \prod U_\alpha$. For such a basis element we have
$$f^{-1}(\prod U_\alpha) = \bigcap f_\alpha^{-1}(U_\alpha).$$
Note that all but finitely many terms in this intersection equal A by definition of U being a basis element. Thus this set is equal to the finite intersection of sets of the form $f_\alpha^{-1}(U_\alpha)$. These sets are all open since each f_α is continuous, so their finite intersection is also open.

- In particular, the map $f : \mathbb{R} \to \mathbb{R}^\omega$ with $f(t) = (t, t, \ldots)$ is continuous under the product topology.

• Aside: the above deals with continuity of (infinite) product spaces, what about sequences in product spaces?

  - E.g. consider $\mathbb{R}^\mathbb{R} = \{f : \mathbb{R} \to \mathbb{R}\}$. Claim that $f_n \to f$ iff $f_n(x) \to f(x)$ for all x, i.e. product topology is the topology of pointwise convergence.
    This is a HW problem.

• Closed sets play nicely with both kinds of product spaces:

Thm: Let $\{X_\alpha\}$ be a family of spaces and $A_\alpha \subseteq X_\alpha$ for all α. If $\prod X_\alpha$ is given either the product or the box topology, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

  - Let $x \in \prod \overline{A}_\alpha$, we want to show $x \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element containing x. Since $x_\alpha \in \overline{A}_\alpha$ and each $U_\alpha \subseteq X_\alpha$ is open, we have that $U_\alpha \cap A_\alpha \neq \emptyset$, and hence $\prod U_\alpha \cap \prod A_\alpha \neq \emptyset$. Since U was arbitrary, it follows that x is in the closure of $\prod A_\alpha$.

  - Now assume $x \in \overline{\prod A_\alpha}$. We want to show $x \in \overline{A_\beta}$ for all β. Fix any neighborhood $U_\beta \subseteq X_\beta$ of x_β. Observe that $U = \prod U_\alpha$ with $U_\alpha = X_\alpha$ for $\alpha \neq \beta$ is a neighborhood of x in the product space, so by hypothesis it intersects $\prod A_\alpha$ at some point y. But then $y_\beta \in U_\beta \cap A_\beta$. This implies every neighborhood of x_β intersects A_β, so $x_\beta \in \overline{A_\beta}$ and hence $x \in \prod \overline{A_\beta}$.

8 Quotient Topology

- Idea: we want to take a space X and "glue" points of X to create a new space.
  - E.g. if you have a square I^2, then gluing two opposite sides gives a cylinder $I \times S^1$.
  - E.g. if you glue the circles of the cylinder you get a torus $S^1 \times S^1$ (same thing happens if you take I^2 and glue both pairs of opposite sides)
- Need to formally define how to "glue" things.

  Aside: There are two different ways of doing this: quotient maps and equivalence relations. The book does the former, we'll mostly be doing the latter (see the supplement on the website).
- Def: an equivalence relation $\sim$ on a set X is a binary relation satisfying reflexivity, symmetry, and transitivity.

  Given $x \in X$, the equivalence class $[x]$ of X is the subset of X with $[x] = \{y \in X : x \sim y\}$. We let $X/\sim$ denote the set of equivalence classes:

  $$(X/\sim) = \{[x] : x \in X\}.$$
- Some examples (Question: what "space" do these equivalence classes define?)
  - Let $X = \mathbb{R}$ and define $\sim$ by $x \sim y$ iff $x - y \in \mathbb{Z}$. The equivalence classes are

    $$[x] = \{\ldots, x-2, x-1, x, x+1, x+2, \ldots\}.$$

    E.g. $[1/2] = \{, \ldots, -1/2, 1/2, 3/2, \ldots\} = [-1/2] = [5/2] = \cdots$.
    In particular we can write

    $$\mathbb{R}/\sim = \{[x] : 0 \le x < 1\} = \{[y] : 99.5 < y \le 100.5\}.$$

    When asking about the space, draw $\mathbb{R}$ as a spiral projecting onto S^1, identify the points of $[0]$ and note that they map to the same thing.
  - Let $A \subseteq X$ be sets, define $\sim$ by $x \sim y$ if $x = y$ or if $x, y \in A$. Equivalence classes are $[x] = \{x\}$ if $x \notin A$ and $[x] = A$ if $x \in A$.

    E.g. if X is a circle and A is two arcs then this turns into a figure eight.

    E.g. if X is a cyllinder and A is one of the faces this turns into a cone.
  - Let $X = [0,1]^2$, define $\sim$ by $(x,0) \sim (x,1)$ for $0 \le x \le 1$ and $(0,y) \sim (1,y)$ (draw diagram of what this represents). Equivalence classes: $[(x,y)] = \{(x,y)\}$ if $0 < x, y < 1$, $[(x,0)] = \{(x,0),(x,1)\}$ if $0 < x < 1$, similar for y, $[(0,0)] = \{(0,0),(1,0),(0,1),(1,1)\}$

- Question: if X is not just a set but a topological space, what's the right way to define open sets on $X/\sim$?

  - Idea: similar to product spaces, there's a natural "projection map" $\pi : X \to X/\sim$ defined by $\pi(x) = [x]$. Just like for products, the "right" notion of topology should be the smallest collection of sets such that π is continuous.

  - To get a handle on this, given $U \subseteq X/\sim$, what is $\pi^{-1}(U)$? Answer: $\bigcup_{[x]\in U}[x]$ (i.e. U is a set of equivalence classes $[x]$, and an element of X maps to an equivalence class of U iff x lies in an equivalence class of U).

  - In particular, if we want π to be continuous, then every open set $U \subseteq X/\sim$ needs to have that $\bigcup_{[x]\in U}[x]$ is open.

- Def: Let X be a topological space and $\sim$ an equivalence relation on X. The quotient topology on $X/\sim$ consists of all sets $U \subseteq (X/\sim)$ such that $\bigcup_{[x]\in U}[x] \subseteq X$ is open in X.

  To emphasize: U is a set of equivalence classes of X, so this union is over subsets of X (and hence does indeed lie in X).

- Prop: the quotient topology is a topology.

  - If $U = \emptyset$ then the union is $\emptyset$ which is empty. If $U = (X/\sim)$ then the union is X.

  - Let $\bigcup_\alpha U_\alpha$ be an arbitrary union of open sets in $X/\sim$. Then

    $$\bigcup_{[x]\in\bigcup U_\alpha}[x] = \bigcup_\alpha \bigcup_{[x]\in U_\alpha}[x],$$

    which is the union of open sets in X by definition of U_α being open.

  - Similarly for a finite intersection

    $$\bigcup_{[x]\in\bigcap U_i}[x] = \bigcap\left(\bigcup_{[x]\in U_i}[x]\right),$$

    which is the finite intersection of open sets.

- Claim: the canonical map $\pi : X \to (X/\sim)$ is continuous with respect to the quotient topology (and the quotient topology is the largest topology for which this holds).

- Problem: how do we show that $X/\sim$ with the quotient topology is homeomorphic to some space Y we care about? This was theoretically the whole reason we're doing all this in the first place.

- Caution: when dealing with maps f from equivalence classes, we have to make sure f is well defined.

E.g. Say $X = \mathbb{R}$ and $x \sim y$ iff $x - y \in \mathbb{Z}$; we intuited that this should be homeomorphic to S^1. Consider the map $f : (\mathbb{R}/\sim) \to S^1$ defined by $f([x]) = (\cos(x), \sin(x))$. What is $f([0])$? $f([1])$? Problem: $[0] = [1]$, so this map isn't well defined.

- We can get around this issue by considering "nice" maps from X and the following.

  Thm (universal property of the quotient topology): let X be a topological space and $\sim$ an equivalence relation on X. Endow $X/\sim$ with the quotient topology and let $\pi : X \to X/\sim$ be the canonical projection.

  Let Y be another topological space and $f : X \to Y$ a continuous function such that $f(x) = f(x')$ whenever $x \sim x'$ in X. Then there exists a unique continuous function $\bar{f} : (X/\sim) \to Y$ such that $f = \bar{f} \circ \pi$.

  - Proof sketch: define $\bar{f}([x]) = f(x)$. This is a well defined map from $(X/\sim)$ to Y, so it remains to show it's continuous.
  - Let $U \subseteq Y$ be open, its preimage $\bar{f}^{-1}(U)$ is open in $X/\sim$ iff the following set is open in X:
  $$\bigcup_{[x] \in \bar{f}^{-1}(U)} [x] = f^{-1}(U),$$
  where the equality used that $x \in f^{-1}(U)$ iff $[x] \subseteq f^{-1}(U)$ (since $f(x) = f(x')$ if $x \sim x'$) which holds iff $[x] \in \bar{f}^{-1}(U)$. This set is open in X since f is continuous.

- Strategy for proving $X/\sim$ is homeomorphic to Y:
  - (1) Find a candidate continuous function $f : X \to Y$.
  - (2) Prove $f(x) = f(x')$ whenever $x \sim x'$; then $\bar{f} : (X/\sim) \to Y$ defined by $\bar{f}([x]) = f(x)$ is well defined and continuous by universal property.
  - (3) Find a candidate inverse continuous function $g : Y \to (X/\sim)$.
  - (4) Prove $f \circ g = id_Y$ and $g \circ f = id_X$.

- Prop: $\mathbb{R}/\sim$ with $x \sim y$ iff $x - y \in \mathbb{Z}$ is homeomorphic to S^1.
  - (1) Find candidate function $f : X \to Y$. Take $f(x) = (\cos(2\pi x), \sin(2\pi x))$. This is continuous from calculus.
  - (2) If $x \sim x'$ then $x - x' \in \mathbb{Z}$, thus

    $f(x) = (\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi x + 2\pi(x' - x)), \sin(2\pi x + 2\pi(x' - x))) = (\cos(2\pi x'), \sin(2\pi x'))$

    Thus the induced map $\bar{f}([x]) = \cos(2\pi x), \sin(2\pi x))$ is well defined and continuous.
  - (3) Construct inverse. We will use the pasting lemma: Let $Y = A \cup B$ with A, B both closed in Y and let $g_1 : A \to Z$ and $g_2 : B \to Z$ be continuous maps that agree

at their intersection. Then the function $g : Y \to Z$ defined by $g(x) = g_1(x)$ for $x \in A$ and $g(x) = g_2(x)$ for $x \in B$ is continuous.

Let $A = \{(x, y) \in S^1 : y \leq 0\}$ and $B = \{(x, y) \in S^1 : y \geq 0\}$. For $z \in A$ there exists a unique $0 \leq x \leq 1/2$ with $z = (\cos(2\pi x), \sin(2\pi x)$, define $g_1' : A \to \mathbb{R}$ by $g_1'(z) = x$. Similarly $z \in B$ can be written uniquely as $(\cos(2\pi x), \sin(2\pi x))$ with $1/2 \leq x \leq 1$. These functions are continuous by calculus but do *not* agree on $A \cap B$ (e.g. they map $(1, 0)$ to 0 and 1).

Can fix this: define $g_1 : A \to (\mathbb{R}/\sim)$ by $g_1 = \pi \circ g_1'$ and similarly define g_2'. Since g_i' and π are continuous, their compositions g_i are also continuous. Moreover, g_1, g_2 agree on their intersection $\{(1, 0), (-1, 0)\}$, so the pasting lemma gives some continuous function $g : S^1 \to (\mathbb{R}/\sim)$.

  - (4) Basic calculations shows $f \circ g$ and $g \circ f$ are identity maps.

- That was a lot, let's take a breather and play with pictures for a little bit.
  - Quotients through diagrams. Idea is that equivalence relations are a pain to write down in full, so often people will just draw pictures to indicate what they mean.
  - Draw square with vertical sides identified. The arrows mean $(0, y) \sim (1, y)$. Physically take a strip of papers with arrows on two sides and glue together; emphasize that this makes it so the arrows line up.
  - Draw previous thing but with one arrow reversed; what equivalence relation does this correspond to? What does this shape look like? Again physically do an example, trace out a line to show it's one-sided.
  - Draw square for torus and ask them what this represents.
  - Draw previous with one pair of arrows reversed; this is called a Klein bottle
  - Draw with both reversed; this is projective space.
- Aside: alternative persepective through quotient maps.
  - Let $q : X \to Y$ be a surjective map. If X is a topolgoical space, then we define the "quotient topology" on Y (with respect to q) by making $U \subseteq Y$ open iff $q^{-1}(U)$ is open in X (note that this is the weakest topology so that q is continuous).
  - Claim: this is a topology.
  - How is this related to previous definition? Given such a q, deifne an equivalence relation $\sim$ on X by having $x \sim x'$ iff $q(x) = q(x')$.

    Claim: $X/\sim$ with the quotient topology is homeomorphic to Y with the "quotient topology".

- The two perspectives (equivalence relations and quotient maps) are entirely equivalent to each other; you should feel free to use whatever makes the most sense.

Post numberphile video on Klein bottles after this lecture.

9 Metric Spaces

9.1 Basics and Examples

- Again want to generalize ideas from real analysis, i.e. from Euclidean topology, so let's take a closer look at this.

- This is the topology generated by the basis of open balls

$$B_n(x,\varepsilon) = \{y \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n} |x_i - y_i|} < \varepsilon\}.$$

  Idea: what if we replaced Euclidean distance with some other form of "distance"?

- Eg say you're in Manhattan and your friend is one block up and to the right from you. How far away are you two?

  - One answer is $\sqrt{2}$ blocks (this is Euclidean distance, i.e. the distance "as the crow flies" since it's only useful if you can ignore the building between you).

  - Another answer is 2 blocks away since that's in practice how far you have to travel.

  - Def: given two points $x, y \in \mathbb{R}^n$, we define the *Manhattan distance* to be $\sum |x_i - y_i|$ (i.e. this is the distance if you can only travel along the axis without cutting corners).

- Broad question: what are other reasonable "distance functions" $d(x, y)$ we can consider between two objects x, y of a set X? In particular, what are reasonable axioms to impose on such a function d?

  - E.g. what should $d(x, x)$ be, i.e. the distance from x to itself? Intuitively this is 0.

  - If you think about things some more you might come up with the following definition.

- Def: given a set X, a function $d : X \times X \to \mathbb{R}$ is a *metric* if (1) $d(x, y) \geq 0$ for all $x, y \in X$ with equality iff $x = y$, (2) $d(x, y) = d(y, x)$ for all $x, y \in X$, and (3) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. Discuss why these are reasonable/cases where maybe these don't quite hold.

- These axioms, in addition to being relatively intuitive, generalize the key properties of the Euclidean distance function, and with these axioms alone we can generalize much of the theory from this case. To do this we need some more definitions analogous to what we had in the Euclidean case.

  - Def: given a metric d, a point $x \in X$, and a real number $\varepsilon > 0$, we define the open ball $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. We will also refer to this as the *ε-neighborhood* of x.

- Claim: the set of open balls is a basis (every point is in at least one. If x is in the intersection of two balls of y, z then you can take the minimum $\varepsilon_y - d(x, y), \varepsilon_z - d(x, z)$ and this will be contained in both by the triangle inequality or something).
- Here we actually proved a useful fact: if x is a ball B, then there exists an $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq B$.
- Def: We define the metric topology (induced by d) is the topology generated by the basis of open balls $B_d(x, \varepsilon)$.

  That is, it is the collection of sets U such that for every $x \in U$ there exists a ball B with $x \in B \subseteq U$.

  By the lemma, this is equivalently the set of U such that every $x \in U$ we have some $B_d(x, \varepsilon) \subseteq U$.

- Examples.
  - For any set X, the function $d(x, y) = 1$ if $x \neq y$ and $d(x, x)$ is a metric (check). What are balls here? Either single points or the whole space. This generates the discrete topology.
  - Claim L2-metric on $\mathbb{R}^n$ is a metric. Balls are balls.
  - Claim L1-metric d_M is a metric. Balls are diamonds.
  - Define square metric on $\mathbb{R}^n$ by $d_s(x, y) = \max\{|x_i - y_i|\}$. Claim metric, only tricky part is triangle inequality which you can do by noting

    $$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i| \leq d_s(x, y) + d_s(x, z).$$

    Balls are squares.

- Question: what do these last three topologies generate? Claim is they're all Euclidean. More generally:

  Thm: let d, d' be metrics on the set X and $\mathcal{T}, \mathcal{T}'$ the topologies they induce. Then $\mathcal{T} \subseteq \mathcal{T}'$ iff for each $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$.

  - (Proof that $\varepsilon - \delta$ implies finer) A set U is open in $\mathcal{T}$ iff for each $x \in U$ we can find a neighborhood $B_d(x, \varepsilon) \subseteq U$ (via that ball centering lemma we proved). Note that every such U also has $B_{d'}(x, \delta) \subseteq U$, so every set open in $\mathcal{T}$ must also be open in $\mathcal{T}'$.
  - Other direction is left as an exercise.

- Claim: Euclidean, square, and Manhattan metric all generate the same topology.
  - Proof (just of Euclidean vs square). Easy to show $d_s(x, y) \leq d_e(x, y) \leq \sqrt{n} d_s(x, y)$. This implies $B_e(x, \varepsilon) \subseteq B_s(x, \varepsilon)$ and that $B_s(x, \varepsilon/\sqrt{n}) \subseteq B_e(x, \varepsilon)$. The previous lemma gives things.

- For Manhattan vs square observe $d_s(x,y) \le d_M(x,y) \le nd_s(x,y)$ and a similar proof works.

9.2 Metrizable Spaces and Properties of Metric Spaces

- Def: we say that a topological space X is metrizable if there exists a metric d on X which induces the topology of X. A pair (X,d) is a metric space if X is a metrizable topology and d is a metric inducing the topology on X.

- Prop: if X is metrizable, then every subspace $A \subseteq X$ is metrizable (one can take the metric for X and restrict it to A; this is still a metric and it gives right topology).

- We've seen $\mathbb{R}^n$ is metrizable for all n. Are infinite product spaces metrizable?

- Thm: let $\bar{d}(a,b) = \min\{|a-b|, 1\}$. For $x, y \in \mathbb{R}^\omega$, define

$$D(x,y) = \sup \frac{\bar{d}(x_i, y_i)}{i}.$$

This is a metric which induces the product topology on $\mathbb{R}^\omega$.

  - Aside: for any metric d, the function $\bar{d}(x,y) := \min\{d(x,y), 1\}$ turns out to be a (bounded) metric which induces the same topology as d, and it can sometimes be convenient to work with this metric rather than d itself.

  - Proof of metric: triangle inequality holds for each term in the sup, so the inequality holds for the sup.

  - Induces product: let $U = \prod U_i$ be a basis element in product topology, say with N a large enough number so that $U_i = \mathbb{R}$ for all $i > N$, and consider some $x \in U$. Since $U_i \subseteq \mathbb{R}$ is open, there exists a $0 < \varepsilon_i \le 1$ with $B(x_i, \varepsilon_i) \subseteq U_i$. Define $\varepsilon = \min_{i \le N} \varepsilon_i / i$, we claim that $B_D(x, \varepsilon) \subseteq U$. Indeed, if y is in this ball then by definition $\varepsilon > D(x,y) \ge \frac{\bar{d}(x,y)}{i}$ for all $i \le N$. Since $\varepsilon \le \varepsilon_i / i$, we have $\bar{d}(x_i, y_i) < \varepsilon_i \le 1$, so $|x_{\alpha_i} - y_{\alpha_i}| < \varepsilon_i$. It follows that $y_i \in U_i$ for all i, proving $y \in U$ as desired.

  - Let $B = B_D(x, \varepsilon)$ with $\varepsilon < 1$, we want to find a basis neighborhood of x in B. Let N be a large enough integer such that $N^{-1} < \varepsilon$. Let V be the basis element with

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \cdots .$$

  Claim that $V \subseteq B$. Indeed, observe that for any $y \in \mathbb{R}^\omega$ and $i \ge N$ we have $\frac{\bar{d}(x_i, y_i)}{i} \le N^{-1}$. Thus

$$D(x,y) \le \max\{\frac{\bar{d}(x_1, y_1)}{1}, \ldots, N^{-1}\}.$$

  If $y \in V$, then this expression is less than ε (since each $\bar{d}$ is at most ε), proving $y \in B_D(x, \varepsilon)$ as desired.

- What about product for other index sets J? Or box topology?
- Idea: to show X is metrizable you just need to construct a metric. To show X *isn't* metrizable, we show that it fails to have some property that every metric space must have.
  - Aside: it turns out you can prove $\mathbb{R}^\omega$ without constructing an explicit metric by using something called the Urysohn metrization theorem (which will be close to the last result of this course).
  - E.g. Prop: every metric space is Hausdorff. Proof
  - Corollary: confinite topology with X infinite is not metrizable.
- (Sequence lemma) Let X be a topological space and $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X is metrizable.
  - Aside: the first part is similar to something we proved earlier (sequence in $A - x$ converging to x means $x \in A'$) and proof is basically the same.
  - Proof 1: Suppose $x_n \to x$ with $x_n \in A$. Then every neighborhood U of x intersects A, so $x \in \overline{A}$ by Theorem 17.5.
  - Proof 2: Suppose X is metrizable, say with d generating its topology and let $x \in \overline{A}$. Consider the balls $B_d(x, 1/n)$. Because $x \in \overline{A}$, there exists a point x_n which intersects A and $B_d(x, 1/n)$. We claim that this sequence converges to x_n. Indeed, any open set U containing x contains an open ball $B_d(x, \varepsilon)$. for $N \geq \varepsilon^{-1}$ we have $x_n \in B_d(x, \varepsilon) \subseteq U$ for all $n \geq N$, proving the claim/result.
  - Note: we didn't really use the full power of the metric space here, only that there exists a "nice" countable family of basis neighborhoods for each point. We will look more at this weaker notion in chapter 4.
- With this we can prove non-metrizability of some topologies on products.

  Prop: The box topology on $\mathbb{R}^\omega$ is not metrizable.
  - $A = \{(x_1, x_2, \ldots,) | x_i > 0 \ \forall i\}$. Claim that $0 = (0, 0, \ldots) \in \overline{A}$. Equivalent to saying every basis element $B = (a_1, b_1) \times \cdots$ containing 0 intersects A. Indeed, $a_i < 0 < b_i$ and hence the point $(\frac{1}{2}b_1, \ldots) \in A \cap B$.
  - Claim no sequence in A converges to 0 let a_n be a sequence with $a_n = (x_{1n}, x_{2n}, \ldots, x_{i,n}, \ldots)$. Since $x_{in} > 0$, we can take the basis element $B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \cdots$. Note that $0 \in B'$ but it contains no a_n point (since $x_{nn} \notin (-x_{nn}, x_{nn})$).

    Aside: this is a somewhat subtle application of a diagonalization argument.
- If J is uncountable then $\mathbb{R}^J$ with the product topology is not metrizable.

- Let $A \subseteq \mathbb{R}^J$ be the points (x_α) with $x_\alpha = 1$ for all but finitely many α. Claim $0 \in \overline{A}$. Take $B = \prod U_\alpha$ a basis element containing 0 and let $\alpha_1, \ldots, \alpha_n$ be the indices with $U_{\alpha_i} \neq \mathbb{R}$. Then the point (x_α) with $x_{\alpha_i} = 0$ and $x_\alpha = 1$ otherwise lies in $A \cap B$.

- Claim no sequence converges. Indeed, let a_n be a sequence and let $J_n \subseteq J$ be the indices with $(a_n)_\alpha \neq 1$. Note that $\bigcup J_n$ is a countable union of finite sets, so there is some $\beta \notin \bigcup J_n$. This means $(a_n)_\beta = 1$ for all n.

  Let $U = \prod U_\alpha$ with $U_\beta = (-1, 1)$ and $U_\alpha = \mathbb{R}$ otherwise. Then no point of a_n is contained in U, so 0 can not be a limit.

9.3 Continuity

- It turns out metric spaces play particularly nicely with continuity. In particular, most definitions from real analysis generalize.

- Let $f : X \to Y$ with X, Y metrizable with metrics d_X, d_Y. f is continuous iff for all $x \in X, \varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$.

  Proof is similar to showing $\varepsilon - \delta$ definition for Euclidean space is equivalent (which they already did for HW).

- Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

  - Assume f is continuous and let $x_n \to x$. We wish to show $f(x_n) \to f(x)$.

  - Let V be a neighborhood of $f(x)$. Continuity means $f^{-1}(V)$ is a neighborhood of x, so for some N we have $x_n \in f^{-1}(V)$ for all $n \geq N$. Thus $f(x_n) \in V$ for $n \geq N$ as well.

  - Now assume X is metrizable and that the convergent condition is satisfied. We aim to show that for any $A \subseteq X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$ (which we proved is equivalent to f being continuous).

  - If $x \in \overline{A}$, then by the sequence lemma there is a sequence of points $x_n \in A$ converging to x. By assumption $f(x_n) \to f(x)$. The other direction of the sequence lemma implies $f(x) \in \overline{f(A)}$, giving $f(\overline{A}) \subseteq \overline{f(A)}$.

  This is where we stopped for the lecture on 10/11/23.

- In metric spaces we can recover other important notions from real analysis.

  Def: let $f_n : X \to Y$ be a sequence of functions with (Y, d) a metric space. We say that (f_n) *converges uniformly* to a function $f : X \to Y$ if for all $\varepsilon > 0$ there exists an N such that $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq N$ and all $x \in X$.

  Warning: the definition depends not only on the topology of Y, but also the specific metric d which induces it.

- Uniform limit theorem: let $f_n : X \to Y$ be a sequence of continuous functions with Y a metric space. If (f_n) converges uniformly to f, then f is continuous.
  - General strategy: try to remember how you proved analogous results from real analysis.
  - Recall that the classic real analysis proof for $X = Y = \mathbb{R}$ goes via a $\varepsilon/3$ argument, i.e. to show f is continuous at x_0 you write

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

  - Let $V \subseteq Y$ be open and $x_0 \in f^{-1}(V)$. We aim to find a neighborhood U of x_0 such that $f(U) \subseteq V$ (this proves that f is continuous at every point x_0 and hence is continuous).
  - Let $y_0 = f(x_0)$ and choose ε so that $B(y_0, \varepsilon) \subseteq V$. Using uniform convergence, we can choose N so that $d(f_n(x), f(x)) < \varepsilon/3$ for all $n \geq N$ and all x.
  - Because f_N is continuous, the set $U = f_N^{-1}(B(f_N(x_0), \varepsilon/3))$ is open in X (Aside: this is a somewhat natural thing to do in the argument because the only thing we know about open sets of X is that they arise as preimages of balls in Y).
  - Note that for all $x \in U$ we have $d(f(x), f_N(x)) < \varepsilon/3$, $d(f_N(x), f_N(x_0)) < \varepsilon/3$ and $d(f_N(x_0), f(x_0)) < \varepsilon/3$. By triangle inequality we find $d(f(x), f(x_0)) < \varepsilon$ for all $x \in U$, proving the result.

- Aside: we saw the product topology, which is induced by the metric $D(x, y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\}$, is the topology of pointwise convergence in that a sequence of functions $f_n : X \to \mathbb{R}$ converges pointwise to some f iff they converge as elements in the product topology.

  Similarly the (perhaps more natural) function $D'(x, y) = \sup\{\bar{d}(x_i, y_i)\}$ is a metric which induces a topology in between the product and box topology called the *uniform topology*. Here $f_n : X \to \mathbb{R}$ converges uniformly to some f iff they converge as elements in the uniform topology.

Possible aside regarding normed vector spaces

Part II

Connectedness and Compactness

10 Connectedness

- We now begin chapter 3, the focus of which is in proving analogs of two very important theorems from calculus: the intermediate value theorem and the extreme value theorem. We begin with the former.

- Intermediate value theorem: if $f : [a, b] \to \mathbb{R}$ is continuous, then for all real r between $f(a)$ and $f(b)$, there exists a $c \in [a, b]$ such that $f(c) = r$.

  While not at all obvious, the reason this theorem works is because the interval $[a, b]$ is "connected". Thus as a starting point we need to generalize "connectedness" to other spaces.

- Intuition: Draw two separate blobs and ask if this space is (intuitively) connected; ask why with an ideal answer being something like you can separate it into two chunks.

  Draw a single blob, asking the same questions; ideal answer is that you can't separate it into two chunks (which is intuitively clear but maybe not so easy to prove).

- Def: let X be a space. We say that X is *disconnected* if there exist non-empty open sets $U, V \subseteq X$ with $U \cap V = \emptyset$ and $U \cup V = X$ (in which case we say U, V is a *separation* of X). We say that X is *connected* if it is not disconnected, i.e. if no separation exists.

- Here's a useful reformulation.

  Prop: a space X is disconnected iff X contains a clopen set $U \neq \emptyset, X$ (if such a set U exists then U, U^c is a separation; if a separation U, V exists then U is clopen).

- Examples

  - The two disconnected blobs is disconnected (proving the other one is connected is substantially harder but is true).
  - Trivial topology is always connected.
  - Discrete topology with at least two elements is always disconnected.
  - $\mathbb{Q}$ is disconnected (take $< \pi$ and $> \pi$)
  - Finite complement topology with $|X| = \infty$ is connected (easiest to see with clopen formulation).

- Thm: the interval $[a, b]$ is connected.

- Note: this is intuitively very obvious, but somewhat tricky to prove.
- Assume for contradiction that U, V is a separation of $[a, b]$, say wlog $a \in U$.
- Define $c = \sup\{x : [a, x] \subseteq U\}$. Note that $c \geq a$.
- Claim: $c \in U$. Else $c \in V$ (since U, V partition $[a, b]$) and hence $c > a$ (since every element of $V \subseteq (a, b]$ is larger than a).

  Because $c > a$ and V is open, there exists $c' < c$ such that $(c', c] \subseteq V$. This implies that for any $c' < c'' < c$ we have $c'' \in V$ and hence $[a, c''] \not\subseteq U$. By definition of c this means $c \leq c''$, a contradiction.
- Claim $c = b$. If $c < b$ then U open means there exists $c < c'$ with $[c, c') \subseteq U$, so any $c'' \in (c, c')$ has $[a, c''] \subseteq U$ which means $c'' \leq c$, a contradiction.
- Since $[a, b] = [a, c] \subseteq U$, we conclude $U = [a, b]$, contradicting $V \neq \emptyset$ and disjoint from U.

That was a lot of work to prove something so simple. Fortunately we can use this result as a black box to prove that many other sets are connected.

- Prop: $\mathbb{R}$, (a, b), and $(a, b]$ are connected.

  Proof: only prove $\mathbb{R}$, other cases are similar. If U, V were a separation, then wlog we can assume $a \in U$ and $b \in V$ with $a < b$. One can check that $U \cap [a, b]$ and $V \cap [a, b]$ must be a separation of $[a, b]$ (since U, V both intersect $[a, b]$), but this contradicts intervals being connected.

- Prop: $[0, 1)$ and S^1 are not homeomorphic (intuitively obvious, but tricky to prove by hand).

  Pf: assume $h : [0, 1) \to S^1$ were a homeomorphism. Let $X = [0, 1) - \{1/2\}$ and $Y = S^1 - \{h(1/2)\}$. One can check $h : X \to Y$ is still a homeomorphism. But X is disconnected (easy to show) and $Y \cong (0, 1)$ is connected, a contradiction by the result above.

- Prop: $\mathbb{R} \ncong \mathbb{R}^n$ for any $n > 1$.

  Pf sketch: $\mathbb{R}$ minus a point is disconnected. $\mathbb{R}^n$ minus a point is homeomorphic to an n-dimensional ball minus its center, and this is homoemorphic to $(0, 1) \times S^{n-1}$ and hence is connected.

  Aside: we actually have $\mathbb{R}^m \ncong \mathbb{R}^n$ for any $m \neq n$, but this is much harder to prove and requires tools from algebraic topology.

10.1 Building Examples and IVT

- We now look at some ways of building new connected sets from old ones.

- Lem: If $\{X_\alpha\}_{\alpha \in A}$ is a collection of connected subspaces of a space Y with $\bigcap_\alpha X_\alpha \neq \emptyset$, then $\bigcup_\alpha X_\alpha$ is connected.

  Pf: Let $p \in \bigcap_\alpha X_\alpha$ and suppose U, V is a separation of Y, say with $p \in U$. Since each X_α is connected, it must lie entirely within either U or V, and since $p \in X_\alpha \cap U$ we must have $X_\alpha \subseteq U$ for all α, and hence $\bigcup X_\alpha \subseteq U$, contradicting $V \neq \emptyset$ being part of a separation.

- Prop: If X, Y are connected, then so is $X \times Y$.

  Proof: Draw picture choose an arbitrary point $(a, b) \in X \times Y$. Note that the horizontal slice $X \times b$ is connected (it's homeomorphic with X) as is $a \times Y$, so $T_a := (X \times b) \cup (a \times Y)$ is connected by the previous lemma, and so is $\bigcup_{a \in x} T_a = X \times Y$.

- Cor: the cubes I^n are connected. This is where we stopped for the lecture on 10/16/23.

- Prop: if X is connected and $f : X \to Y$ is continuous and surjective, then Y is connected. (Pre-image of a separation for Y is a separation for X).

  Here or earlier note that if X is connected then so is any space Y which is homeomorphic to X (i.e. connectedness is a topological property).

- Cor: If X is connected, then every quotient space $X/\sim$ is connected (the projection map is continuous and surjective).

- Cor: the spheres S^n for $n \geq 1$ are connected (draw this for $n = 1, 2$ using I^1, I^2).

- Thm (IVT): if $f : X \to \mathbb{R}$ is continuous and X is connected, then for all $a < c < b$ with $a, b \in f(X)$, there exists $x \in X$ such that $f(x) = c$.

  Remark: this generalizes classic IVT since we now know intervals are connected.

  Pf: if not $X = f^{-1}(-\infty, c) \cup f^{-1}(c, \infty)$ would be a separation.

- Aside: one can prove a stronger version of this result where you replace $\mathbb{R}$ with any "ordered space".

- Cor: if $f : S^2 \to \mathbb{R}$ is continuous then there exists $x_0 \in S^2$ such that $f(x) = f(-x)$.

  Pf: Define $g : S^2 \to \mathbb{R}$ by $g(x) = f(x) - f(-x)$. Note that g is continuous because f is.

  Let $x \in S^2$ be arbitrary and $a = g(x)$. If $a = 0$ then we're done, otherwise $g(-x) = -g(x) = -a$, so $a, -a \in f(S^2)$. Because S^2 is connected, there exists $x_0 \in S^2$ with $g(x_0) = 0$, i.e. with $f(x_0) = f(-x_0)$.

- Corollary: there exist two opposite points on the earth with the exact same temperature (can let $f : S^2 \to \mathbb{R}$ represent the temperature of each point on the earth).

- Aside: using results form algebraic topology one can in fact show that if you have two continuous functions $f, g : S^2 \to \mathbb{R}$, then there exists x_0 with $f(x_0) = f(-x_0)$ *and* $g(x_0) = g(-x_0)$.

10.2 Variants of Connectedness

- Here we look at some variants of connectedness.

- Intuitively, if a space X is connected then it should be possible to "walk" between any two points of X. Draw a squiggle between two points, note that this is basically a map from an interval into the space.

  Def: If $x, y \in X$, then a *path* from x to y is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. We say X is *path connected* if every pair of points of X can be joined by a path in X.

- Q: how does connectedness and path connectedness relate to each other? Are they equivalent? Does one imply the other?

- Prop: if X is path connected then X is connected (assume have separation U, V let $u \in U, v \in V$ and $f : [a, b] \to X$ a path between them. Then $f^{-1}(U), f^{-1}(V)$ is a separation of $[a, b]$).

- Does the converse hold? No but it's not easy to show.

  Let $S = \{(x, \sin(1/x) : 0 < x \leq 1\} \subseteq \mathbb{R}^2$ draw this, basically start a sin curve from the right and get more and more compressed as you tend to the origin, and define the *topologist's sine curve* $\overline{S} \subseteq \mathbb{R}^2$.

  Thm: $\overline{S}$ is connected but not path connected.

  - Here and below the following will be useful: we say a set $A \subseteq X$ is connected if A with the subspace topology is connected.

  - Intuition (proof is too complex): $\overline{S}$ consists of S together with part of the y-axis.

  - For connected, general fact is that if $A \subseteq X$ is a connected subspace, then so is $\overline{A}$ (because $\overline{A} \setminus A$ is "arbitrarily close" to A). S (being the image of a connected set $(0, 1]$) is connected, thus so is $\overline{S}$.

  - For path-connected, claim is there's no path from the y-axis to any point on the other curve. Point is that although y-axis and S are arbitraily close, there is a gap and you can't quite jump into it.

  - Full proof is not hard but somewhat complex, see the book.

- Aside: there are "local" versions of these concepts that are sometimes useful.

  Def: a set X is *locally connected* at x if for every neighborhood U of x, there is a connected neighborhood V of x contained in U (meaning V with the subspace topology is connected), and X is *locally connected* if it is locally connected at each point.

  The definition for X being *locally path connected* if completely analogous.

- Remark: X is locally connected iff there exists a basis $\mathcal{B}$ of X where each $B \in \mathcal{B}$ is connected.

- It turns out that e.g. connectedness and local connectedness are incomparable properties.

  Make 2x2 grid labeled with connected/local connected and yes/no. YY=$\mathbb{R}$, intervals, S^n. NY=two disjoint intervals. YN=Topologist's sine curve. NN=$\mathbb{Q}$.

- Sames for path-connected. YY=$\mathbb{R}, I, S^n$. NY=disjoint union of intervals, YN=modified sine curve (draw a line from the right of the sine curve below everything and then connecting to the middle of the y-axis. NN=$\mathbb{Q}$.

This is where we stopped for the lecture on 10/18/23.

11 Compactness

- We used connectedness to generalize IVT, now we want to generalize EVT.

  Recall EVT: if $f : [a,b] \to \mathbb{R}$ is continuous, then there exist $c, d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ (i.e. f has a max/min).

- We now want to define some property of $[a,b]$ which makes the EVT go through (similar to how we defined connectedness to make IVT hold).

  While the definition of connectedness was relatively intuitive, this will not be the case for EVT (and indeed it took many years and wrong failed attempts to get this correct).

- Intuitively Underline three times the property we want is that X is closed and bounded. The real property is the following (which we'll go over slowly):

  Def: a collection of sets $\mathcal{A}$ is a *cover* of a set X if $X \subseteq \bigcup_{A \in \mathcal{A}} A$. It is called an *open covering* if each $A \in \mathcal{A}$ is open in X. We say that a subset $\mathcal{A}' \subseteq \mathcal{A}$ is a *finite subcover* of $\mathcal{A}$ if $\mathcal{A}'$ is finite and also a cover of X.

  A space X is *compact* if every open cover $\mathcal{A}$ of X contains a finite sub-cover.

- To emphasize, this is a weird definition that should not make any sense to you right now.

  Let's look at some examples and non-examples.

  - Let X be any space and $\mathcal{A} = \{X\}$. Is $\mathcal{A}$ a cover? Does $\mathcal{A}$ have a finite subcover? Does that mean X is compact?

    Emph: to prove X is compact, you have to make an argument about **all** covers $\mathcal{A}$ of X.

  - Let X be a topology on a finite number of points. Is X compact? Yes: let $\mathcal{A}$ be a cover, by def each $x \in X$ has $x \in A_x$ for some $A_x \in \mathcal{A}$, $\{A_x\}_{x \in X}$ is a finite subcover of $\mathcal{A}$.

    Alt proof: X has only finitely many open sets (this is a simpler but less "robust" proof than the one above).

  - Let X have the finite complement topology. Is X compact? Yes, basically the same proof after fixing some $A \in \mathcal{A}$.

  - Let X have the trivial topology. Is X compact? Yes, e.g. because there's only finitely many open sets in the topology.

  - Let X have the discrete topology. Is X compact if $|X| = \infty$? No, take $\mathcal{A} = \{\{x\} : x \in X\}$, this has no finite subcover.

  - Is $\mathbb{R}$ compact? No, take $\mathcal{A} = \{(-n, n) : n \in \mathbb{Z}_{>0}\}$ (e.g. for any finite subset there exists some M larger than every element of $\mathcal{A}'$).

    Morally: $\mathbb{R}$ isn't compact because it isn't bounded.

- Is $(0,1)$ compact? No because it's homeomorphic to $\mathbb{R}$.

  Is $(0,1]$ compact? No take $\mathcal{A} = \{(1/n, 1] : n \in \mathbb{Z}\}$, any finite collection is always bounded away from 0.

  Morally: $(0,1]$ isn't compact because it isn't closed.

- Is $[0,1]$ compact? Yes but it's not easy to show and we're going to postpone it.

- Let $X = \{0\} \cup \{1/n : n \in \mathbb{Z}_{\geq 1}\}$ draw this. Is X compact? Yes: for any $\mathcal{A}$ let $A_0 \in \mathcal{A}$ be an element containing 0. This contains all but finitely many points of X (since it contains some ball centered at 0), each of which can be dealt with by picking some $A_x \in \mathcal{A}$ containing it.

  Morally: X is compact because it is closed and bounded (again, this should always be your intuition, but it will fail for weird spaces).

- Let's explore some ways of creating new compact sets from old ones; many of which will be analogous to statements for connected sets. We first establish a lemma that will be very useful when working with subsets.

  Def: if X is a space, we say that a set $Y \subseteq X$ is compact if Y with the subspace topology is compact.

  Lem: a subset $Y \subseteq X$ is compact iff every covering $\mathcal{A}$ of Y by open sets in X contains a finite subset $\mathcal{A}' \subseteq \mathcal{A}$ which covers Y.

  - Assume Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then $\mathcal{A}_Y = \{A_\alpha \cap Y\}$ is a covering of Y by open sets of Y, so there exists some finite subcover $\{A_{\alpha_1} \cap Y, \ldots,\}$, and hence $\{A_{\alpha_1}, \ldots\} \subseteq \mathcal{A}$ is a finite cover of Y.

  - (Sketch) Assume the other condition holds and let $\mathcal{A}'$ be a cover of Y. Since each $A'_\alpha \in \mathcal{A}$ is open in Y, this means there exists an open set $A_\alpha \subseteq X$ such that $A'_\alpha = A_\alpha \cap Y$. One can find a finite subcollection of the A_α by assumption, which projects to a finite subcollection of the A'_α.

    The notation in this proof isn't great, probably change to e.g. $\mathcal{A}_Y$.

- Prop: if $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

  Pf: Let $\mathcal{A}$ be a cover of $f(X)$ by open sets of Y. Then $\{f^{-1}(A) : A \in \mathcal{A}\}$ is an open cover of X. Since X is compact, there exists a finite subcover $f^{-1}(A_1), \ldots, f^{-1}(A_n)$. Then the sets $A_1, \ldots, A_n$ are a finite subcover for Y.

  This is where we stopped for the lecture on 10/30/23.

- Cor: If $X \cong Y$ then X is compact iff Y is (i.e. compactness is a topological property).

- Thm: Let X be compact. If $Y \subseteq X$ is closed then Y is compact. The converse holds if X is Hausdorff (i.e. the only compact subspaces of X are closed).

– Assume $Y \subseteq X$ is closed. We'll use the previous lemma. Let $\mathcal{A}$ by a cover of Y using open sets of X. We want to use that X is compact to get that we can find a finite cover; so can we turn $\mathcal{A}$ into a cover for X somehow?

  Let $\mathcal{B} = \mathcal{A} \cup \{X - Y\}$ which is an open cover of X since Y is closed. Since X is compact there exists a finite subcover $\mathcal{B}' \subseteq \mathcal{B}$, and hence $\mathcal{B}' \cap \mathcal{A}$ is a finite subcover of $\mathcal{A}$ for Y, proving the result.

– Assume X is Hausdorff and that $Y \subseteq X$ is compact. We want to show that Y is closed, i.e. that every $x \in X - Y$ has a neighborhood $x \in U$ disjoint from Y. Motivating example: what do we do when $Y = \{y\}$ (which is closed because Hausdorff)?

  By Hausdorff property, for every $y \in Y$ there exist disjoint neighborhoods U_y, V_y for x, y. Observe that $\{V_y\}_{y \in Y}$ is a cover of Y using open sets of X. Since Y is compact, one can find a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Note that $\bigcap U_{y_i}$ is disjoint from $Y \subseteq \bigcup V_{y_i}$ and open (since its the finite intersection of open sets), so this gives the desired neighborhood.

Rmk: the condition that X be Hausdorff is necessary for converse to hold: in the finite complement topology (which isn't Hausdorff), every set is compact whether they be open or not.

- Neat corollary: if $f : X \to Y$ is a continuous bijection, and if X is compact and Y is Hausdorff,then f is a homoemorphism.

  Pf: To prove f^{-1} is continuous, it suffices to prove images of closed sets of X are closed in Y. Every closed $A \subseteq X$ is compact by the previous theorem, so $f(A)$ is compact by the previous proposition. Since Y is Hausdorff, $f(A)$ is closed by the previous theorem.

- Def: if (X, d) is a metric space, we say that $Y \subseteq X$ is *bounded* if there exists $r \in \mathbb{R}_{>0}$ such that $d(x, y) \leq r$ for all $x, y \in Y$.

  Prop: if (X, d) is a metric space, then the only compact subspaces $Y \subseteq X$ are bounded (sketch: if not, then for any $x \in Y$ the balls $B_d(x, n)$ with $n \in \mathbb{Z}_{\geq 1}$ are a cover without a finite subcover).

- Cor: if (X, d) is a metric space, then the only compact subspaces $Y \subseteq X$ are closed and bounded.

  Very important fact: the converse is true for Euclidean space.

- The key here will be proving things for intervals (similar to what happened with connected).

  Thm: the interval $[0, 1]$ is compact.

  – Let $\mathcal{A}$ be an open cover of $[0, 1]$, $c = \sup\{0 \leq x \leq 1 : [0, x] \text{ is covered by a finite subfamily of } \mathcal{A}\}$, and $A \in \mathcal{A}$ a set containing c.

- Claim: $c = 1$. If $c = 0$, then $[0, \varepsilon] \subseteq A$ for some $\varepsilon > 0$, which means $c \geq \varepsilon$, a contradiction.

  If $0 < c < 1$, then $[c - \varepsilon, c + \varepsilon] \subseteq A$ for some $\varepsilon > 0$. By definition of c, the interval $[0, c - \varepsilon]$ has a finite subcover from $\mathcal{A}$, and adding A to this gives a finite subcover to $[0, c + \varepsilon]$, so $c \geq c + \varepsilon$, a contradiction.

- Because $c = 1$, we have $[1 - \varepsilon, 1] \subseteq A$ for some $\varepsilon > 0$. Since $[0, 1 - \varepsilon]$ has a finite subcover, so does $[0, 1]$, proving the result.

- Can boost this result to higher dimensions.

  Thm: If X, Y are compact, then $X \times Y$ is compact.

  For this we'll need a lemma.

- Tube lemma: let X, Y be spaces with Y compact, and let $N \subseteq X \times Y$ be an open set containing the "slice" $\{x_0\} \times Y$. Then N contains a "tube" $W \times Y$ where $W \subseteq X$ is a neighborhood of x_0 Draw picture.

  - Possible way to motivate the proof: know every open set is the union of basis elements. Simplest case is that it's (contained in) union of finitely many basis elements (which we can assume contain x), in which case the solution is to just take the intersection of the horizontal sets. Thus suffices to reduce to this finite cover, and for this it makes sense to use compactness.

  - Pf: Since N is open, it can be written as the union of basis elements $U \times V$. Since $\{x_0\} \times Y$ is homeomorphic to Y it is compact, so it can be covered by finitely many of these basis elements $U_1 \times V_1, \ldots, U_n \times V_n$. Wlog we can assume $x_0 \in U_i$ for all i (otherwise we can throw out this set).

  - Let $W = \bigcap U_i$, which is an open neighborhood of x_0. We claim that the $U_i \times V_i$ sets cover $W \times Y$. Indeed let $(x, y) \in W \times Y$. By assumption of these sets covering $\{x_0\} \times Y$, there exists some i such that $y \in V_i$, and hence $(x, y) \in U_i \times V_i$ since $x \in \bigcap U_j$.
    Since $W \times Y \subseteq \bigcup U_i \times V_i \subseteq N$, we conclude the result.

- Now we can prove $X \times Y$ is compact if X, Y are both compact.

  - Pf: let X, Y be compact and $\mathcal{A}$ an open cover of $X \times Y$.

    Claim: for each $x \in X$, there exists a neighborhood $W \subseteq X$ such that $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. Indeed, since $\{x\} \times Y \cong Y$ is compact, there exists a finite subcover $A_1, \ldots, A_n$ of this slice, so taking $N = \bigcup A_i$ and applying the tube lemma gives the result.

  - The sets W_x are an open cover of X, so by compactness there exists some finite subcover $W_1, \ldots, W_m$. The union of the tubes $W_i \times Y$ covers all of $X \times Y$, and since each of these finitely many tubes can be covered with finitely many sets, we conclude the result.

- Remark: this implies finite products of compact spaces are compact, what about infinite spaces?

  Tychonoff Thoerem: arbitrary products of compact spaces are compact in the product topology (This is not an easy result: it requires the axiom of choice and is the central focus of Chapter 5).

- Heine Borel Thm: $X \subseteq \mathbb{R}^n$ is compact iff X is closed and bounded.

  Already saw that if X is compact then it must be closed and bounded (since $\mathbb{R}^n$ is a metric space).

  If X is closed and bounded, then $X \subseteq [-M, M]^n$ for some sufficiently large M. This is compact by the previous theorem, and thus $X \subseteq M$ (which is a closed subspace of a compact set) is compact.

  Wow, amazing!

- Note: Heine-Borel does not hold for arbitrary metric spaces (e.g. not for the discrete metric).

- Cor: $S^1 \ncong [0, 1)$ (former is compact, other is not).

11.1 EVT and Consequences

- Now that we have a good handle on what compact sets look like, we can state the generalized version of EVT.

  Extreme Value Theorem: if $X \neq \emptyset$ is compact and $f : X \to \mathbb{R}$ is continuous, then $f(X) \subseteq \mathbb{R}$ has a maximum and a minimum.

  Pf: Assume $f(X)$ has no maximum, i.e. for all $z \in f(X)$ there exists some $y \in f(X)$ with $z < y$. This implies that $\mathcal{A} = \{f^{-1}(\infty, y) : y \in f(X)\}$ is an open cover of X. Note that $\mathcal{A}$ has no finite subcover (as this would imply $f^{-1}(\infty, y)$ is a subcover for some y), contradicting X being compact.

- We can use this to prove an analog of the uniform convergence theorem, which we recall says that continuous maps $f : [0, 1] \to \mathbb{R}$ are uniformly continuous. For this we need some definitions/lemmas.

  Def: if $B \neq \emptyset$ is a subset of a metric space (X, d), then the *diameter* of B is $\sup\{d(a, b) : a, b \in B\}$.

  Lebesgue number lemma: let $\mathcal{A}$ be an open covering of a metric space (X, d). If X is compact, then there is a $\delta > 0$ such that for every subset $B \subseteq X$ of diameter less than δ there exists some $A \in \mathcal{A}$ containing B. The number δ is called the *Lebesgue number* for the covering $\mathcal{A}$.

I.e. this says that all sufficiently small sets are contained in some element of $\mathcal{A}$.

- Useful definition: for $C \neq \emptyset$ and $x \in X$ we define the *distance* from x to B by $d(x, C) := \inf\{d(x, y) : y \in C\}$ (note: it is not hard to show this is a continuous function).

- If $X \in \mathcal{A}$ then this holds for all δ, so assume this is not the case.

- Let $A_1, \ldots, A_n$ be a finite subcover of X maybe note intuition that working with this smaller cover can only make our lives harder since the enemy could have given this at the start and let $C_i := X - A_i$ (which are non-empty by assumption). Define $f : X \to \mathbb{R}$ by $f(x) = n^{-1} \sum d(x, C_i)$ (i.e. this is the average distance from x to one of the C_i sets).

- Note that f is continuous, so by EVT it has a minimum value δ, i.e. every $x \in X$ have average distance at least δ to the C_i. We want to show this δ works.

- Claim: $f(x) > 0$ for all x (and hence $\delta > 0$). Indeed, $x \in A_i$ for some i, which means there exists some ε-neighborhood of x in A_i. This means $d(x, C_i) \geq \varepsilon$ and $f(x) \geq \varepsilon/n$.

- Let B be a subset of diameter less than δ and choose $x_0 \in B$. By the diameter condition, B lies in the δ-neighborhood of x_0. Note $\delta \leq f(x_0) \leq \max_i d(x_0, C_i)$. If the maximum is achieved by i then the δ-neighborhood of x_0 is contained in $X - C_i = A_i$, so $B \subseteq A_i$.

Rmk: this result turns out to be used a fair amount in algebraic topology, and we'll also use it to show various forms of compactness are equivalent to each other.

- Def: a function f from a metric space (X, d_X) to (Y, d_Y) is said to be *uniformly continuous* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x_0, x_1 \in X$ with $d_X(x_0, x_1) < \delta$ we have $d_Y(f(x_0), f(x_1))$.

  Uniform continuity theorem: if $f : (X, d_X) \to (Y, d_Y)$ is continuous and X is compact, then f is uniformly continuous.

  - Given $\varepsilon > 0$, let $\mathcal{A}'$ be the open cover of Y by balls $B(y, \varepsilon/2)$, and let $\mathcal{A} = \{f^{-1}(A') : A' \in \mathcal{A}'\}$. Let δ be the Lebesgue number of $\mathcal{A}$.

  - For any $x_0, x_1 \in X$ with $d_X(x_0, x_1) < \delta$, the set $B = \{x_0, x_1\}$ has diameter less than δ, so there exists some preimage of a ball containing B, i.e. $\{f(x_0), f(x_1)\} \subseteq B(y, \varepsilon/2)$ for some y. This means $d(f(x_0), f(x_1)) < \varepsilon$ as desired.

12 Variants of Compactness

As noted before, there were many alternative definitions proposed for compactness during the development of topology, many of which still have their uses.

- Def: a space X is said to be *limit point compact* if every infinite subset $A \subseteq X$ has a limit point, i.e. a point x such that every neighborhood of x intersects A in some point *other than itself.*

  Thm: if X is compact, then it is limit point compact.

  - Let X be compact and assume for contradiction that there exists $A \subseteq X$ infinite with no limit point. This implies that A contains all its limit points (trivially), so A is closed. Since X is compact, this implies A is also compact.

    Goal is to find an open cover without a finite subcover.

  - Because each $a \in A$ is not a limit point, there exists a neighborhood U_a with $U_a \cap A = \{a\}$. Now $\mathcal{U} = \{U_a : a \in A\}$ is a cover of A with no finite subcover (since A is infinite and each element of U_a intersects A in one vertex), contradicting A being compact.

- The converse does not hold.

  Claim: if $X = \{a, b\}$ has the trivial topology, then $X \times \mathbb{Z}_{>0}$ is LPC but not compact. What topology are we putting on $\mathbb{Z}_{>0}$ here? Is it discrete?

  - LPC: every $A \neq \emptyset$ has a limit point (e.g. if $(a, n) \in A$ then every neighborhood of (b, n) intersects A).

  - Not compact: the open cover with sets $\{a, b\} \times \{n\}$ has no finite subcover.

- Def: a space X is said to be *sequentially compact* if every sequence of points has a convergent subsequence.

  E.g. $X = [0, 1]$ is sequentially compact (since every sequence in X is bounded via Bolzano-Weierstrass).

- In general sequential compactness is incomparable to compactness (though the examples are a little complex). However, all of these notions of compactness agree for metrizable spaces.

  Thm: Let X be a metrizable space. The following are equivalent: (1) X is compact, (2) LPC, (3) sequentially compact.

  - Fix some metric d inducing X. Already showed (1) implies (2).

  - (2) implies (3): let (x_n) be a sequence in X; how should we construct a set of infinite points A?

    Take $A = \{x_n : n \in \mathbb{Z}_{>0}\}$. Is this infinite? Not necessarily.

If A is finite, then some $x \in A$ has $x = x_n$ for infinitely many n; the subsequence using such x_n converges.

Thus we may assume A is infinite, so by LPC there exists some limit point $x \in A$.

Aside: at this point the sequence lemma almost gives us what we want, but we have to be a bit more careful with the argument.

Fact: if x is the limit point of a set $A \subseteq X$ and X is metrizable, then every neighborhood of x intersects A in infinitely many points (in fact this holds for Hausdorff spaces in general).

Build our convergent subsequence as follows. Choose any $x_{n_1} \in A \cap B(x, 1)$, then choose any $x_{n_2} \in A \cap B(x, 1/2)$ with $n_2 > n_1$ (can do this because the intersection is infinite), and iteratively continue in this way with $x_{n_t} \in A \cap B(x, 1/t)$ and $n_t > n_{t-1}$. Easy to check this is a convergent subsequence, proving the result.

- (3) implies (1). We first solve a special case:

  Claim 1: for all $\varepsilon > 0$, there exists a finite cover of X by ε-balls (equivalently, the cover $\mathcal{A}$ consisting of all ε-balls has a finite subcover).

  Suppose not Draw a picture for this. Choose any $x_1 \in X$, then $x_2 \in X - B(x, \varepsilon)$, and so on choosing $x_n \in X - \bigcup_{i<n} B(x_i, \varepsilon)$. This defines an infinite sequence (else there'd be a finite covering), so by assumption there exists some subsequence x_{n_k} converging to some y. However, the ball $B(y, \varepsilon/3)$ contains at most one x_{n_k} (since they're all at least ε away from each other), contradicting it converging to y.

- Claim 2: the Lebesgue number lemma holds, i.e. for all open covers $\mathcal{A}$ there is a $\delta > 0$ such that every for every subset $B \subseteq X$ of diameter less than δ there exists some $A \in \mathcal{A}$ containing B. (Again we know this must in particular hold if X is compact).

  Assume this failed for some $\mathcal{A}$. This means for all n, there exists non-empty $C_n \subseteq X$ such that $diam(C_n) < 1/n$ but $C_n \nsubseteq A$ for any $A \in \mathcal{A}$.

  Choose some $x_n \in C_n$ for each n. This has a convergent subsequence $x_{n_k} \to y$, and there exists some $A \in \mathcal{A}$ with $y \in A$. Because A is open, $B(y, \varepsilon) \subseteq A$ for some $\varepsilon > 0$. But because $x_{n_k} \to y$, there exists some K such that $x_{n_k} \in B(y, \varepsilon/3)$ for all $k \geq K$. For some $k \geq K$ we have $diam(C_{n_k}) < \varepsilon/3$, which implies $C_{n_k} \subseteq B(y, \varepsilon) \subseteq A$, a contradiction.

- Now we can finally prove things. Let $\mathcal{A}$ be an open cover of X and let $\delta > 0$ be the Lebesgue number of this cover (exists by claim 2). By Claim 1 there exists a finite cover $\mathcal{B}$ of X by open balls of radius $\delta/3$ (and diameter $2\delta/3 < \delta$). This means for each $B_i \in \mathcal{B}$ there is some $B_i \subseteq A_i \in \mathcal{A}$. Since the B_i are a finite cover for X, the A_i do as well, proving the result.

• Aside: as noted, it is not true in general that X is compact iff every sequence has a convergent subsequence. However, it turns out that X is compact iff every "generalized sequence" has a convergent "generalized subsequence".

  These "generalized subsequences" are called nets; see the supplementary exercises at the end of chapter 3.

13 Local Compactness

- Obs: $\mathbb{R}$ is "almost" sequentially compact in that x_n has a convergent subsequence provided its bounded. As such, $\mathbb{R}$ will become sequentially compact if we add a "point at infinity" draw it wrapping around to a common ∞.

  Here we look at other spaces that are "almost compact" and how to adjust them to be genuinely compact.

- Def: a space X is *locally compact* at $x \in X$ if there is a compact subspace $C \subseteq X$ that contains a neighborhood of x. We say X is *locally compact* if it is locally compact at every point.

  Note: while it uses similar language, this definition is very different from the one used to define local connectedness. In particular, being compact automatically implies being locally compact (which was NOT true for local connected)

  Aside: Many theorems (though not the ones we've seen in this course) which are true for compact spaces are also true for locally compact ones (eg existence of Haar measure)

- Examples:

  - $\mathbb{R}$ is locally compact.
  - $\mathbb{R}^n$ is locally compact.
  - $\mathbb{Q}$ is not locally compact Maybe prove; see commented out comments
  - If X is discrete then X is locally compact (via taking $C = U = \{x\}$).

- Thm: a space X is locally compact and Hausdorff iff there exists a compact Hausdorff space Y satisfying the following (1) X is a subspace of Y, (2) $Y - X$ has a single point ∞ (the point at infinity).

  Moreover, if Y, Y' are two spaces satisfying these conditions, then there is a (unique) homeomorphism between Y, Y' which equals the identity map on X.

- Discussion:

  - That is, every "reasonable" locally compact space can be made compact by adding an extra point to it, and moreover there's essentially a unique way to add this point.
  - Eg if $X = \mathbb{R}$, what is Y?
  - Eg if $X = S^1$ (which is in particular locally compact), what is Y? Ans: $S^1 \sqcup \{\infty\}$ draw picture

- Proof: existence of Y implies X locally compact and Hausdorff.

- Since $X \subseteq Y$ and Y is Hausdorff we have that X is Hausdorff.
- For local compactness, consider any $x \in X$. Because Y is Hausdorff, there exist disjoint neighborhoods U, V of x, ∞. Note that $C = Y - V$ is closed in Y and hence compact (since Y is compact). It is also a compact subspace of X, so $U \subseteq C \subseteq X$ shows X is locally compact at x, proving this result.

- Proof: uniqueness.
  - Say $Y = X \cup \{\infty\}$ and $Y' = X \cup \{\infty'\}$. What map should we use as our homeomorphism?
    $h(x) = x$ for $x \in X$ and $h(\infty) = \infty$.
  - This is a bijection, and by symmetry it suffices to show h^{-1} is continuous, i.e. that $h(U) \subseteq Y'$ is open whenever $U \subseteq Y$ is open.
  - If $U \subseteq X$ is open in Y, then U is also open in X. Note that the singleton $Y' - X$ is closed since Y is Hausdorff, so X is open in Y' and hence U is also open in Y'. Because $h(U) = U$, we see that h maps this open set of Y to an open set of Y'.
  - Now consider $U \subseteq Y$ open with $\infty \in U$. Then $C = Y - U$ is closed in Y and hence compact. This means $C \subseteq X$ is a compact subspace of X, and because $X \subseteq Y'$ is a subspace, this means $C \subseteq Y'$ is a compact subspace of Y'. Because Y' is Hausdorff, this means C is closed in Y', and hence $Y' - C = h(U)$ is open in Y'.

- Finally we prove that X locally compact and Hausdorff implies the existence of such a Y.
  - The previous proof suggests how to define a topology on $X \cup \{\infty\}$: let $\mathcal{T}$ consist of all sets (1) $U \subseteq X$ which are open in X, and (2) $Y - C$ where C is a compact subspace of X [Note that the previous proof implies that any topology we could consider would have to be a subset of $\mathcal{T}$].
  - Claim: this is a topology. $\emptyset, Y$ are of types (1),(2).
    Intersection of two (1) sets is still (1) (intersection of open sets is still open). Intersection of two (2) sets is (2) (the union of two compact sets is also compact). $U \cap (Y - C) = U \cap (X - C)$; since X is Hausdorff the compact subset C must be closed, so this is of type (1).
    Arbitrary of unions of (1) is still (1). Arbitrary union of (2) is of the form $Y - \bigcap C_\alpha$; note that $\bigcap C_\alpha$ is closed in X (since each is closed and intersection of closed is closed) and a subset of a compact set C_β, so this intersection is compact and of type (2). Note $U \cup (Y - C) = Y - (C - U)$; $C - U$ is closed in X and subset of compact C, so this is compact and of type (2).
  - Claim $X \subseteq Y$ is a subspace. Every $U \subseteq X$ open in X is open in Y by construction (so the topology of X is contained in the subspace topology). If V is a neighborhood

of ∞ in Y, then $Y - V$ is compact in X, which by Hausdorff means its closed, which means $V \cap X$ is open in X.

In particular this claim makes talking about "open sets" easier throughout.

  - Claim Y is compact. Let $\mathcal{A}$ be an open cover of Y and take $A \in \mathcal{A}$ containing ∞. By definition of the topology on Y, $C := Y - A$ is compact, so one can find a finite subcover $\mathcal{A}' \subseteq \mathcal{A}$ of C so $\mathcal{A}' \cup \{A\}$ is a finite subcover of Y.

  - Claim Y is Hausdorff. Let $x, y \in Y$ distinct. If $x, y \in X$ then X being Hausdorff gives the desired open sets (which are still open in Y). If $y = \infty$, then by local compactness at x, there exists $x \in U \subseteq C$ and now $V = Y - C$ is a neighborhood of y disjoint from U.

  - Since $Y - X$ has a single point by construction, we're done.

- Prop: let X, Y be as in the theorem. If X is not compact, then $\overline{X} = Y$.

  Pf: suffices to show $X \subseteq Y$ is not closed. Indeed if it were then X would be compact, which we assumed not to be the case.

- Aside: in general if $X \subseteq Y$, then we say that Y is a *compactification* of X if Y is compact Hausdorff and $\overline{X} = Y$. If X, Y are as in the theorem (and X isn't compact already), then we say Y is **the** one-point compactification.

  This one-point compactification is in some sense the "smallest" compact set containing X. One can also consider the "largest" one, which is known as the Stone-Cech compactification (see chapter 5).

Probably add in Theorem 29.2 and its corollaries.

Part III

Countability and Separation Axioms

We now shift away from properties motivated by calculus and instead turn to "purely topological" properties.

14 Countability Axioms

- Def: We say that a space X has a *countable basis at* $x \in X$ if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that every neighborhood U of x contains some $B \in \mathcal{B}$. We say that X is *first-countable* if it has a countable basis at x for all $x \in X$.
- Main example: metric spaces via $B_d(x, 1/n)$.

  In fact, many theorems we proved for metric spaces hold more generally for first-countable spaces (with the exact same proofs).

  Thm: Let X be space.

  (a) Let $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. The converse holds if X is first-countable.

  (b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$. The converse holds if X is first-countable.
- (More important) Def: if X has a countable basis $\mathcal{B}$, then we say that X is *second-countable* (for convenience we will sometimes just say X has a countable basis).
- Obs: second-countability implies first-countability.

  E.g.

  - $\mathbb{R}$ (take intervals with rational endpoints).
  - $\mathbb{R}^n$ (products of intervals as above)
  - $\mathbb{R}^\omega$ with product topology ($\prod U_\alpha$ where finitely many are open intervals with rational endpoints and the rest are $\mathbb{R}$)
  - $\mathbb{R}^J$ with J uncountable is not second-countable (in fact, it isn't even first-countable because as we showed in chapter 2 it fails to satisfy the sequence lemma).
  - If X has the discrete topology then it is second-countable iff $|X|$ is countable. It is always first countable (since it's metrizable).
- Countability plays nicely with subspaces and products.

Thm (a) Subspaces of second-countable spaces are second-countable. (b) The countable product of second-countable spaces is second-countable.

This same result holds with "second-countable" replaced by "first countable.

- For subspace $A \subseteq X$, $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis.

  If $\mathcal{B}_i$ is a basis for X_i, then the collection of sets $\prod U_i$ with $U_i \in \mathcal{B}_i$ for finitely many i and $U_i = X_i$ for all others is a basis.

- First-countability proof is similar.

* Thm: if X has a countable basis $\mathcal{B} = \{B_1, \ldots, \}$, then the following holds: (a) every open cover of X contains a countable subcover (i.e. we're almost compact), (b) X contains a countable subset $A \subseteq X$ such that $\overline{A} = X$.

  Note: a set A with $\overline{A} = X$ is called *dense*, the canonical example being $\mathbb{Q} \subseteq \mathbb{R}$.

  - (a): let $\mathcal{A}$ be an open cover and $I \subseteq \mathbb{Z}_+$ be the set of integers n such that there exists $A_n \in \mathcal{A}$ with $B_n \subseteq A_n$.

    Claim that $\{A_i : i \in I\}$ is a (countable) subcover. Indeed, pick any $x \in X$ and $A \in \mathcal{A}$ containing x. Since $\mathcal{B}$ is a basis there exists some $x \in B_n \subseteq A$, so $n \in I$ and hence A_n exists and $x \in A_n$.

  - (b): For each (non-empty) B_n, pick an arbitrary $x_n \in B_n$ and let D be the set of these points.

    Claim: D is (countable) dense subset. Indeed, pick any y and U a neighborhood. There is some $B_n \subseteq U$, this intersects D at x_n.

  Aside: these two consequences of second-countability are quite important and are sometimes taken as separate axioms: any X satisfying (a) is called Lindelöf, and any X satisfying (b) is said to be separable (unrelated to what we're about to talk about).

15 Separation Axioms

- Recall def of Hausdorff (can find disjoint neighborhoods around distinct points).

  We've seen already that this is a really useful condition to have. Here we look at strengthening of this condition (some of which we've already seen implicitly).

- Def: Let X be a space such that one-point sets are closed. X is said to be *regular* if for every $x \in X$ and closed $B \subseteq X$ disjoint from x, there exist disjoint open sets U, V containing x, B. We say that X is *normal* if for every pair $A, B \subseteq X$ of disjoint closed sets there exist disjoint open sets U, V containing A, B. Draw pictures.

- Remark: Normal implies regular implies Hausdorff (we need to make one-point sets being closed part of the definition in order for this go through).

- Let's look at some general classes of spaces which satisfy these conditions.

  (Essentially already proved) Lem: compact Hausdorff spaces are regular.

  Sketch: take any $x \in X$, $B \subseteq X$ closed (which means compact). For each $y \in B$ take U_y, V_y disjoint neigborhoods of x, y. Because B is compact you can find a finite cover $V_{y_1}, \ldots, V_{y_n}$, the union of these and intersection of the U_{y_i} give the desired sets.

- (In Exercise 26.5 you essentially proved) Thm: compact Hausdorff spaces are normal.

  Sketch: take any $A, B \subseteq X$ disjoint. For each $x \in A$ we know by previous thing that there exist disjoint U_x, V_x containing x, B. Again can find finite cover $U_{x_1}, \ldots, U_{x_n}$ because A is compact, and the same trick as before works.

- Thm: Every metrizable space X is normal.

  - Proof: let d be a metric inducing X and A, B disjoint closed sets. For each $a \in A$, let ε_a be such that $B(a, \varepsilon_a)$ is disjoint from B, and similarly define ε_b (these exist because e.g. $a \notin B$ and B is closed).

  - Take $U = \bigcup B(a, \varepsilon_a/2)$ and $V = \bigcup B(b, \varepsilon_b/2)$.
    Claim: these are disjoint (any $z \in B(a, \varepsilon_a/2) \cap B(b, \varepsilon_b/2)$ implies $d(a, b) < (\varepsilon_a + \varepsilon_b)/2 \leq \max\{\varepsilon_a, \varepsilon_b\}$, contradicting one of a, b being distance at least $\varepsilon_a, \varepsilon_b$ from B, A).

- Here's a "non-example": define $\mathbb{R}_K$ to be the topology on $\mathbb{R}$ generated by the basis of open intervals (a, b) and sets of the form $(a, b) - K$ where $K := \{\frac{1}{n} : n \in \mathbb{Z}_+\}$.

  Claim: $\mathbb{R}_K$ is Hausdorff but not regular. Hausdorff is easy (take two disjoint intervals containing the points). For not regular it suffices to look at $0, K$ (but the details are a little fiddly).

- Rmk: there also exist spaces X which are regular but not normal, but these are a little fiddly to describe. Specifically, we take X to be the Sorgenfrey plane $\mathbb{R}_\ell^2$ where $\mathbb{R}_\ell$ is the lower limit topology. Alternatively $\mathbb{R}^J$ with J uncountable, but this is hard to prove.

- Here's a useful reformulation of the definitions:

  Lem: Let X be a space where one-point sets are closed. (a) X is regular iff for any $x \in X$ and neighborhood U of x, there is a neighborhood V of x with $\overline{V} \subseteq U$. (b) X is normal iff for any closed $A \subseteq X$ and open set U containing A, there is an open set V containing A with $\overline{V} \subseteq U$.

  - Assume X is regular and consider some $x \in U \subseteq X$. How do you find a closed set? Then $B = X - U$ is closed, so there exist disjoint open sets V, W containing x, B.
    Note that $\overline{V}$ is disjoint from W (and hence B) since any $y \in W$ has a neighborhood disjoint from V (namely W).

  - Converse: consider any $x \in X$ and closed $B \subseteq X$ disjoint from x. Let $U = X - B$ and V the neighborhood of x guaranteed. Then $W = X - \overline{V}$ gives the result.

  - Proof for (b) is essentially identical.

- Thm: (a) Subspaces of Hausdorff spaces are Hausdorff; arbitrary products of Hausdorff spaces are Hausdorff.

  (b) Subspaces of regular spaces are regular; arbitrary products of regular spaces are regular.

  - (a) Subspaces: assume $Y \subseteq X$ with X Hausdorff. For any $x, y \in Y$, there exist disjoint neighborhoods $U, V \subseteq X$, and $U \cap Y, V \cap Y$ does the job.

  - (a) Products: let $X = \prod X_\alpha$ with each X_α Hausdorff and consider $x, y \in X$. If $x \neq y$ then there exists some β with $x_\beta \neq y_\beta$, let U_β, V_β be disjoint neighborhoods in X_β and take $U = \prod U_\alpha$ with $U_\alpha = X_\alpha$ for $\alpha \neq \beta$ and similarly define $V = \prod V_\alpha$. These are disjoint and open.

  - (a) Subspaces: assume $Y \subseteq X$ with X regular and consider $x \in Y$ and $B \subseteq Y$ closed in Y and disjoint from x. Go through the naive argument from above, then ask where went wrong. Issue is that B is not necessarily closed in X.
    One can show that if $\overline{B}$ is the closure of B in X, then $\overline{B} \cap Y = B$ (and in particular $x \notin \overline{B}$ which is all we really need). Thus by regularity of X we can find disjoint U, V for $x, \overline{B}$ and then $U \cap Y, V \cap Y$ work.

  - (b) Products: let X_α be a family of regular spaces and $X = \prod X_\alpha$. Recall that we proved in general that $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$. In particular, one-point sets are closed in X because they are closed in each X_α.
    We use the reformulation of regularity above. Let $x \in X$ and U a neighborhood of x, and let $\prod U_\alpha$ be a basis element containing x and contained in U.

Since each X_α is regular, there exist neighborhoods $x_\alpha \in V_\alpha$ with $\overline{V_\alpha} \subseteq U_\alpha$ (and if $U_\alpha = X_\alpha$, we choose $V_\alpha = X_\alpha$). This implies

$$\prod U_\alpha \supseteq \prod \overline{V_\alpha} = \overline{\prod V_\alpha},$$

where this last step holds because we proved $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$ in general for product spaces. Thus $\prod V_\alpha$ is the desired neighborhood (noting that this is open since $V_\alpha \neq X_\alpha$ for only finitely many α by construction).

- Remark: subspaces and products of normal spaces are NOT normal in general.

  E.g. if J is uncountable, then $\mathbb{R}^J$ is not normal (but the above shows it is regular).

- The following result will be crucial for proving the Urysohn Metrization theorem (i.e. the whole point of this chapter).

  Thm: if X is regular and has a countable basis $\mathcal{B} = \{W_1, \ldots, \}$ (i.e. is second-countable), then X is normal.

  - Again we use the reformulation of regularity.

    For each $a \in A$, there exists a neighborhood U_a such that $\overline{U_a}$ is disjoint from B. These U_a form a cover of A. Because X is second-countable so is A, and hence this open cover has a countable sub-cover $\{U_1, \ldots, \}$. Similarly there exists a countable cover $\{V_1, \ldots, \}$ for B such that $\overline{V_n} \cap A = \emptyset$.

  - Naive attempt: take $U = \bigcup U_n$ and $V = \bigcup V_n$. Issue: there's no way to guarantee U, V are disjoint from each other.

    Patch: define $U_n' = U_n - \bigcup_{i \leq n} \overline{V_i}$ (which is open) and similarly $V_n' = V_n - \bigcup_{i \leq n} \overline{U_i}$, and let $U' = \bigcup U_n'$ and $V' = \bigcup V_n'$.

    Observe that $A \subseteq U'$ (since in particular $U_n' \cap A = U_n \cap A$), $B \subseteq V'$, and $U' \cap V' = \emptyset$ (note that $U_m' \cap V_n'$ with e.g. $m \leq n$ is empty by construction since we removed $U_m' \subseteq \overline{U_m'}$ from V_n').

16 Urysohn

16.1 Urysohn Lemma

Possibly omit talking about completely regular spaces if you think you can pack the metrization theorem in here as well.

- We now prove what the book refers to as the first "deep" result in this course which is very useful in topology and analysis.

  Urysohn lemma: let X be a normal space and A, B disjoint closed subsets of X. Then there exists a continuous map $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. Draw a picture with A, B a disjoint union of closed intervals in $\mathbb{R}$.

  "In normal spaces you can separate closed sets by continuous functions."

- Proof idea

  - Draw a smallish open set U_0 around A disjoint from B define $f(x) = 0$ if $x \in U_0$ and $f(x) = 1$ otherwise. This is very far from continuous.

    Draw another set $U_{1/2}$ around U_0 and disjoint from B. Define $f(x) = 0$ if $x \in U_0$, $f(x) = 1/2$ if $x \in U_{1/2} \backslash U_0$, and $f(x) = 1$ otherwise. This is also far from continuous, but its better than the previous example.

    Idea now is to further refine these U_r sets for every rational number $r \in \mathbb{Q} \cap [0, 1]$ so that the "space" between these U_r sets become arbitrarily small.

- Actual proof:

  - Let $R = \mathbb{Q} \cap [0, 1]$.

    (Motivated by the above) Goal: find open sets U_r for each $r \in R$ such that whenever $r < r'$, we have $\overline{U}_r \subseteq U_{r'}$ (i.e. these sets are nested in a reasonable way).

  - Since R is countable, we can write it as $\{r_0, r_1, r_2, \ldots\}$ with $r_0 = 0$ and $r_1 = 1$. Define $U_1 = U_{r_1} = X - B$. Because $A \subseteq U_1$ is closed, by normality of U_0 there exists $U_0 \supseteq A_0$ with $\overline{U_0} \subseteq U_1$.

  - Suppose we have constructed sets U_{r_i} satisfying the goal for all $i < n$. Consider r_n and let $p = \max\{r_i : i < n,\ r_i < r_n\}$ and $q = \min\{r_i : i < n,\ r_i > r_n\}$ (that is p/q is the immediate predecessor/successor of r_n in $\{r_0, \ldots, r_{n-1}\}$). Note that these max/mins are over non-empty sets (since $r_0 = 0, r_1 = 1$ have already been dealt with), so these are well defined.

    Observe that $\overline{U_p}$ is a closed set and $U_q \supseteq \overline{U_p}$. By normality, there exists some open set U_{r_n} such that $\overline{U_p} \subseteq U_{r_n}$ and $\overline{U_r} \subseteq U_q$.

    Claim: for all $i < n$, we have $\overline{U_{r_n}} \subseteq U_{r_i}$ if $r_i > r_n$ and $\overline{U_{r_i}} \subseteq U_{r_n}$ if $r_i < r_n$ (in the second case we inductively have $\overline{U_{r_i}} \subseteq \overline{U_p} \subseteq U_{r_n}$, other direction is similar).

- Induction gives sets U_r with the properties of Goal. Define $f : X \to [0,1]$ by $f(x) = \inf\{r : x \in U_r\}$ if $x \in U_1$ and $f(x) = 1$ if $x \notin U_1$.

  This has $f(x) = 0$ for $x \in A$ since $A \subseteq U_0$, and $f(x) = 1$ for $x \in B$ since $U_1 = X - B$. It remains to show this is continuous.

- Consider some $x \in X$ and let $y = f(x) \in [0,1]$. Assume $0 < y < 1$. For any $\varepsilon > 0$, we need to show that x has a neighborhood U with $f(U) \subseteq (y - \varepsilon, y + \varepsilon)$. Let r, s be rationals such that $y - \varepsilon < r < y < s < y + \varepsilon$. Observe that $x \in U_s - \overline{U_r}$ by definition of $f(x) = y$ and that $f(U_s - \overline{U_r}) \subseteq [r, s] \subseteq (y - \varepsilon, y + \varepsilon)$, so f is continuous at x.

  The cases $f(x) = 0, f(x) = 1$ are similar.

16.2 Urysohn Metrization Theorem

- (Recall a long time back we asked) Question: given a space X, is it metrizable?

  We saw a few special cases of the question above. However, there exist a number of general theorems giving necessary and/or sufficient conditions for this. One of which is the following.

  Urysohn metrization theorem: every regular space X with a countable basis is metrizable. Note somewhere (at least in the proof) that this implies X is normal

- Corollary: $\mathbb{R}^\omega$ is metrizable (recall before that we did this explicitly by considering $\sup\{\bar{d}(x_i, y_i)/i\}$).

- Remark: the converse does not hold (via taking discrete topology on an uncountable set X).

- Proof Ideas:

  - Recall $F : X \to Y$ is an *imbedding* if F restricts to a homeomorphism from X to $f(X)$.

  - Idea: subspaces of metrizable spaces are metrizable. Thus it suffices to find an imbedding $F : X \to Y$ where Y is metrizable. Moreover, it makes sense to pick a Y which is "infinite dimensional" as otherwise it might not be possible to imbed X if X itself has "infinite dimension".

  - Does anyone remember an infinite dimensional metric space? $\mathbb{R}^\omega$ with product topolgy, so let's try this.

    Note that $F : X \to \mathbb{R}^\omega$ can be expressed as $F(x) = (f_1(x), f_2(x) \ldots,)$ for functions $f_n : X \to \mathbb{R}$. Thus it suffices to find a "nice" sequence of functions f_n.

- Claim: There exists a **countable** collection of continuous functions $f_n : X \to [0,1]$ such that for any $x \in X$ and neighborhood U of x, there exists an N such that $f_N(x) > 0$ and $f_N(X - U) = 0$.

- Warmup: show that given any x, U there exists a function f with $f(x) > 0$ and $f(X - U) = 0$. Ans: Urysohn applied to $A = \{x\}, B = X - U$ (note that we proved regular and second countable implies X is normal).

  This is a good start, but the number of pairs (x, U) is typically uncountable, so we'll need to somehow get away with fewer functions.

- Let $B_1, \ldots$ be a countable basis for X and let $\mathcal{A} = \{(B_m, B_n) : \overline{B_m} \subseteq B_n\}$. Note that $\mathcal{A}$ is countable.

  By Urysohn, for each $(B_m, B_n) \in \mathcal{A}$ there exists continuous $f_{m,n} : X \to [0,1]$ such that $f(\overline{B_m}) = 1$ and $f_{m,n}(X - B_n) = 0$.

  For any $x \in X$ and neighborhood of U, by definition of a basis there exists some B_n with $x \in B_n \subseteq U$. By regularity, there exists B_m with $x \in B_m$ and $\overline{B_m} \subseteq B_n$. Thus $(B_m, B_n) \in \mathcal{A}$ and $f_{m,n}(x) > 0$ and $f_{m,n}(X - U) = 0$ as desired.

• Finishing the proof.

  - Define $F : X \to \mathbb{R}^\omega$ by $F(x) = (f_1(x), \ldots)$ with f_n as above. We want to show this is an imbedding.

  - F is continuous because each f_n is continuous and since $\mathbb{R}^\omega$ has ht eproduct topology.

    F is injective: if $x \neq y$ then by regularity there is a neighborhood U containing x but not y, so for some n we have $f_n(x) > 0$ and $f_n(y) = 0$ and hence $F(x) \neq F(y)$.

  - Remains to show that F is an open map, i.e. that if $U \subseteq X$ is open then $F(U) \subseteq F(X)$ is open in $F(X)$ (in the subspace topology from $\mathbb{R}^\omega$).

  - Fix some $x \in U$ and let $z = F(x) \in F(U)$. Let N be such that $f_N(x) > 0$ and $f_N(X - U) = 0$.

    Let $W = f(X) \cap \pi_N^{-1}(0, \infty) = f(X) \cap (\mathbb{R}, \mathbb{R}, \ldots, \mathbb{R}, (0, \infty), \mathbb{R}, \ldots)$. Note that W in the subspace topology of $F(X)$ (since intersection with open set in $\mathbb{R}^\omega$). Further, $z \in W \subseteq F(U)$. As each $z \in F(U)$ has a neighborhood in $F(U)$, this set must be open. Thus $F(U)$ is open, completing the proof.

16.3 Related Concepts

• We saw that Urysohn lemma was really useful, so let's go back and reframe it slightly.

  - Def: we say that two sets $A, B \subseteq X$ can be separated by a continuous function if there exists a continuous $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

    Urysohn lemma: in a normal space, disjoint closed sets can be separated by continuous functions.

  - Remark: it is not true that in regular spaces you can separate points x from disjoint closed sets $B \subseteq X$.

Why does the proof fail? Again you can define $U_1 = X - B$ and then U_0 an open set around a with $\overline{U_0} \subseteq U_1$. But now you want to find e.g. $U_{1/2}$ containing $\overline{U_0}$ with closure in U_1, and for this you really need normality.

- Motivated by the above:

  Def: a space X is *completely regular* if one-point sets are closed and if one can separate points $x \in X$ from disjoint closed sets $B \subseteq X$.

- Urysohn shows that every normal space is completely regular. Completely regular spaces are regular (take $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$ to be your open sets). Thus this condition fits in between normality and regularity.

- Remark: some (older) texts refer to Hausdorff, regular, and normal spaces as T_2, T_3, and T_4 spaces, with there also existing T_0, T_1, T_5, and T_6. Some people refer to complete regularity as $T_{3\ 1/2}$ since it lies in between T_3 and T_4.

- Prop: subspaces of completely regular spaces are completely regular; arbitrary products of completely regular spaces are completely regular.

- Urysohn metrization gives sufficient conditions (which are not necessary because of discrete topologies); can we weaken the conditions to get an if and only if?

  - Def: we say a family of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ is *locally finite* if each $x \in X$ is contained in finitely many $A \in \mathcal{A}$.

    E.g. if $X = \mathbb{R}$ and $\mathcal{A} = \{(n, n+2) : n \in \mathbb{Z}\}$ then this is locally finite.

  - Nagata-Smirnov metrization theorem: a space X is metrizable iff X is regular and there exists a basis $\mathcal{B}$ such that there is a countable union $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite.

    I.e. X is regular and has a countably locally finite basis.

  - E.g. If X has a countable basis (like in Urysohn) then we can take $\mathcal{B}_n = \{B_n\}$.

  - E.g. If X has the discrete topology then $\mathcal{B}_1 = \{\{x\} : x \in X\}$ is locally finite.